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ON THE GROWTH OF GRAVITY-CAPILLARY WAVES BY WIND.

Klaartje van Gastel

In May 1983 this study was submitted as a master's thesis to the Department of Theoretical Physics of the University of Utrecht. Supervisors were P.A.E.M. Janssen and G.J. Komen, both of the Royal Netherlands Meteorological Institute, and Th. Ruijgrok, of the University of Utrecht.

Abstract

For the case of gravity-capillary waves ($\lambda \sim 1$ cm) the exact equations for the wave motion in air and water are derived. Effects of viscosity and surface tension are included. A closed expression for the growth-rate of the waves is found by expanding the equations in $\delta = \frac{\rho_a}{\rho_w}$ and R_w, the Reynolds number in water. This expression is compared to the results of Miles. The expression still contains an unknown perturbation stream-function in air. This streamfunction is determined for the case of a linear wind profile. In this linear wind profile (LWP) model it is a linear combination of an exponential function and a weighted integral of the Airy function. By estimating the integral with the use of an expansion in $R_a^{2/3}$ growth is shown to be possible (eq. V.62). The range of validity of V.62 is investigated. It is valid for a fairly small wavenumber interval. In this interval, however, there is good agreement with numerical results of Kawai which in turn have been confirmed experimentally.

CONTENTS

I. Introduction

Consider the case of no wind and a perfectly calm sea. Suddenly the wind rises. Though the watersurface looked flat probably some extremely small disturbances were present. These can be seen as waves with infinitesimally small amplitudes. These waves, however small, influence the air flow. The changed flow in its turn has an effect on the infinitesimal waves. For certain wavenumbers these reciprocal influences may be such that the corresponding waves grow; the sea no longer appears smooth but ripples are seen.

The situation described above is the basic idea of the mechanism of instability of shear flow. The adjective shear is used to indicate that the wind speed depends on the height. Recently considerable evidence has been collected to show that indeed the instability mechanism can be used to describe wave growth in its initial stages (Valenzuela, '76, Kawai, '79).

The study of instability of shear flows dates from the end of the nineteenth century. Miles in 1957 was the first to seriously introduce stability analysis as a way of describing wave growth. In an inviscid approach to the problem he found that the curvature at the critical height - the height where the air speed equals the wave speed - is the essential feature of the wind profile. If no curvature at the critical height exists no instability is found.

This implies that only larger waves ($\lambda \gtrsim 30$ cm) could grow directly due to the wind. This follows from the typical wind profile. Very close to the watersurface the wind increases linearly with height (thus no curvature), from $\pm$ 0,5 mm onwards the increase is logarithmic. Roughly speaking, for waves with wavelength of 30 cm or less the critical height is within the linear part of the profile, so these waves could never grow. This is in contradiction with every-day experience and with measurements by Kawai ('79).

In this paper the equations governing the growth of gravity-capillary waves are solved. In this way the contradiction mentioned above in helped out of this world. In chapter II a mathematical description of fluid motion is given. From the general case the equations specific for small amplitude waves on an interface of two fluids are derived. These are the Orr-Sommerfeld equation and the continuity conditions at the interface. In chapter III a theoretical argument is given for the windprofile in the atmosphere up to a height of 10 m. Alsc attention is paid to the flow in the water. Chapter IV contains a historical review of the use of the instability theory to describe the growth of waves. It centers round the theory of Miles mentioned above.

The next chapters contain original research. In chapter V I solve the viscid Orr-Sommerfeld equations plus boundary conditions for the case of gravity-capillary waves ($\lambda \sim 1$ cm). The phase velocity is found (V.37) and an expression for the growth-rate (V.41), still containing the unknown streamfunction in air. For the special case of a linear wind profile - the LWP model - the streamfunction in air is solved. In the LWP model the growth is approximately determined (V.62). In chapter VI I discuss various aspects of the theory developed in chapter V. It turns out that the apparent contradiction mentioned above does not exist as Miles' theory is valid only when viscosity can be neglected. The LWP model does predict growth. The range of validity of the expressions V.37, 41 and 62 is considered. The values for the phase velocity and for the growth-rate obtained from the mentioned expressions are compared to numerical values. Another topic is the term dominating the transfer of energy from wind to waves. Chaper VII contains the conclusions I drew from the foregoing chapters. An important conclusion is that viscosity is responsible for the instability found in the LWP model. This possible effect of viscosity is not widely recognized, though Reynolds already speculated about it.

To describe the waves at the watersurface and the mechanism for their growth knowledge is necessary of the motions of the fluid everywhere in water and air. This knowledge can be found in the theory of fluid dynamics. In the present chapter the complete set of equations governing fluid flow is recapitulated. First the equations are formulated as to be valid for any fluid motion. Then step by step it is shown how they can be made more specific for the case of waves on an interface of air and water. The final results is the Orr-Sommerfeld equation plus boundary conditions.

The equations are expressed in 6 variables: $\bar{u}$, the velocity, p, the pressure, ρ, the density, and T, the temperature. All the variables depend on $\bar{x}$, the location in space, and t, the time. The complete set of equations governing fluid flow is given by five conservation laws and the ecuation of state (cf. Batchelor '81, p. 164):

conservation of mass: $\frac{1}{\rho}\frac{D\rho}{Dt} + \bar{\nabla}.\bar{u} = 0$

conservation of momentum:

$$\rho\frac{Du_i}{Dt} = \rho F_i - \frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_j}\left\{2\mu\left(e_{ij} - \tfrac{1}{3}\Delta\delta_{ij}\right)\right\}$$

$$e_{ij} = \tfrac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \qquad \text{II.1}$$

$$\Delta = e_{ii} = \bar{\nabla}.\bar{u}$$

μ = the viscosity $\qquad$ $\bar{F}$ = body force per unit mass of fluid

conservation of energy: $T\frac{DS}{Dt} = \Phi + \frac{1}{\rho}\frac{\partial}{\partial x_i}\left(k_H\frac{\partial T}{\partial x_i}\right)$

S = entropy per unit mass

k_H = thermal conductivity

$\Phi = \frac{2\mu}{\rho}\left(e_{ij}e_{ij} - \frac{1}{3}\Delta^2\right)$

Effects of damping have been ignored in the equation for conservation of energy. ρ and T are chosen as the two parameters of state. μ and k_H are regarded as known functions of them.
The equation of state depends on the nature of the fluid.

equation of state: $f(\rho, p, T) = 0$

In addition to these equations there are boundary conditions to be fulfilled.

For the present case the following assumptions are justified (Batchelor, '81, p. 174):

- the fluid is incompressible and no other effects are present that could cause the density to vary during its evolution. Also the density is the same everywhere in the fluid.
- the temperature is constant in time and space.

Due to these assumptions the equations of state and of conservation of energy become irrelevant and are supplemented by

$$\rho(\bar{x}, t) = \rho \qquad\qquad T(\bar{x}, t) = T$$

Two other assumptions are

- μ is constant in time and space (this follows from ρ and T being constant).
- the only body force present is the gravitational force $\bar{g}$.

The equations now read:

$$\bar{\nabla} \cdot \bar{u} = 0 \qquad \text{II.2}$$

$$\rho \frac{D\bar{u}}{Dt} = \rho \bar{g} - \bar{\nabla} p + \mu \nabla^2 \bar{u} \qquad \text{II.3}$$

The first of these is called the continuity equation; the second is the Navier-Stokes equation. The boundary conditions cannot be specified yet; they still depend on, for instance, the geometry of space occuped by the fluid.

In the present case equations II.2 and II.3 have to be solved for air and for water in such a way that the solutions are matched correctly at the free interface and that the boundary conditions are satisfied. To this end the next assumptions are that turbulence and other nonlinear features of the flow can be neglected in both air and water. This is justified in part as the main interest will lie in a region very close to the interface and in this region the flow is laminar (Miles, '59). Also Valenzuela ('76) notes that as the instability in shear flow, which is a linear effect, can account for the measured growth-rates of gravity-capillary waves there is no need to introduce other features, which mistify the essentials and bring along a lot of cumbersome work.

To describe the situation of a large sea and air above it I take the water to occupy the half infinite region

$$y \leq \eta(x,z,t)$$

y is taken to be the vertical and η is the interface. The position of the interface still has to be determined. The air is taken to fill the rest of space. All quantities are supposed uniform in the z-direction and the argument z will be dropped from now on.

The boundary conditions at the interface are (Batchelor, '81, p. 148-150):

kinematical condition: $\frac{D\eta}{Dt} = v$ on $y = \eta(x,t)$ II.4

u = velocity in x -direction

v = velocity in y-direction

From the kinematical condition the form of the free surface can be derived and the continuity of the normal component of the velocity. The tangential component of the velocity is continuous due to viscosity (cf. Batchelor, '81, p. 148-149):

$$u_a = u_w \qquad \text{on } y = \eta(x,t) \qquad \text{II.5}$$
$$v_a = v_w$$

the suffix a stands for air; w for water.
continuity of shearing stress:

$$\mu_a e_{ija} t_i n_j = \mu_w e_{ijw} t_i n_j \qquad \text{on } y = \eta(x,t) \qquad \text{II.6}$$

the tensor e_{ij} is defined by II.1, $\bar{t}$ is a vector tangential to the surface and $\bar{n}$ is a vector normal to the surface.

Continuity of normal stress:

$$p_a - 2\rho_a\nu_a\,(e_{ija}n_i n_j) = p_w - 2\rho_w\nu_w e_{ijw}n_i n_j + T\,\frac{1}{R}$$

$$\text{on} \quad y = \eta\,(x,\ t) \qquad \text{II.7}$$

R is the local radius of the curvature of the surface, T is the coefficient for the surface tension and $\nu = \frac{\mu}{\rho}$ is the kinematical viscosity.

A further simplification is an expansion of all the quantities in a small parameter. As small parameter ε, the wave steepness, is taken. The zeroth order or basic flows are independent of the waves. They are the wind and the wind-induced surface current. The waves are small perturbations of these flows. The amplitude of the waves is supposed small enough to justify only first order expansions being used. If the waves grow the basic flows are no longer stable.

The basic flows are taken to be simple shearing motions; in chapter III the exact profiles are given. They are indicated with capital letters and directed in the horizontal plane:

$$U_a = U_a\,(y) \qquad V_a = 0 \qquad y \geqslant \eta$$

$$U_w = U_w\,(y) \qquad V_w = 0 \qquad y \leqslant \eta$$

The interface is taken to be flat when no waves are present:

$$H\,(x,t) = 0$$

It can be verified that for these flows it is possible to satisfy II.2 - II.7. The conditions at the interface become:

$$U_a(o) = U_w(o)$$

$$\mu_a U'_a(o) = \mu_w U'_w(o) \qquad \text{II.8}$$

$$P_a(o) = P_w(o)$$

The equations II.2 and II.3 and the conditions at the interface are linearized around these basic flows to obtain equations governing the wave-like perturbations. To this end all quantities are explicitly expanded:

$$u_i = U_i + \varepsilon u_i^{(1)} \qquad i = a \text{ or } w$$

$$v_i = \varepsilon v_i^{(1)}$$

$$P_i = P_i + \varepsilon p_i^{(1)} \qquad \text{II.9}$$

$$\eta = H + \varepsilon \eta^{(1)}$$

The linearized forms of the Navier-Stokes equations are

$$\frac{\partial u^{(1)}}{\partial t} + U \frac{\partial u^{(1)}}{\partial x} + v^{(1)} \frac{\partial U}{\partial y} = -\frac{1}{\rho} \frac{\partial p^{(1)}}{\partial x} + \nu \left(\frac{\partial^2 u^{(1)}}{\partial x^2} + \frac{\partial^2 u^{(1)}}{\partial y^2} \right) \qquad \text{II.10}$$

$$\frac{\partial v^{(1)}}{\partial t} + U \frac{\partial v^{(1)}}{\partial x} = -\frac{1}{\rho} \frac{\partial p^{(1)}}{\partial y} + \nu \left(\frac{\partial^2 v^{(1)}}{\partial x^2} + \frac{\partial^2 v^{(1)}}{\partial y^2} \right) \qquad \text{II.11}$$

The continuity equation becomes

$$\frac{\partial}{\partial x} u^{(1)} + \frac{\partial}{\partial y} v^{(1)} = 0 \qquad \text{II.12}$$

The conditions at the interface are

$$\frac{\partial \eta^{(1)}}{\partial t} + U \frac{\partial \eta^{(1)}}{\partial x} = v^{(1)} \qquad \text{on } y = 0 \qquad \text{II.13}$$

$$v_a^{(1)} = v_w^{(1)} \qquad u_a^{(1)} = u_w^{(1)} \qquad \text{on } y = 0 \qquad \text{II.14}$$

$$\mu_a \left[\frac{\partial u_a^{(1)}}{\partial y} + \frac{\partial v_a^{(1)}}{\partial x} + \eta^{(1)} \frac{\partial^2 U_a}{\partial y^2} \right] = \mu_w \left[\frac{\partial u_w^{(1)}}{\partial y} + \frac{\partial v_w^{(1)}}{\partial x} + \eta^{(1)} \frac{\partial^2 U_w}{\partial y^2} \right] \qquad \text{II.15}$$

on $y=o$

$$\eta^{(1)} \frac{\partial P_a}{\partial y} + p_a^{(1)} - 2\mu_a \left[-\frac{\partial \eta^{(1)}}{\partial x} \frac{\partial U_a}{\partial y} + \frac{\partial v_a^{(1)}}{\partial y} \right] = \eta^{(1)} \frac{\partial P_w}{\partial y} + p_w^{(1)} +$$
$$- 2\mu_w \left[-\frac{\partial \eta^{(1)}}{\partial x} \frac{\partial U_w}{\partial y} + \frac{\partial v_w^{(1)}}{\partial y} \right] + T \frac{\partial^2 \eta^{(1)}}{\partial x^2} \qquad \text{on } y=o \qquad \text{II.16}$$

At infinite depth and height the disturbances are taken to vanish:

$$p^{(1)} = u^{(1)} = v^{(1)} = 0 \qquad y = \pm \infty \qquad \text{II.17}$$

When it is not indicated whether a quantity is to be taken in air or water it may be either.

The final step to get the form of the differential equations in which they can be solved is to introduce a streamfunction (cf. Batchelor, '81 , p. 74).

This function is defined by

$$u^{(1)} = \frac{\partial \psi}{\partial y}$$

$$v^{(1)} = -\frac{\partial \psi}{\partial x} \qquad \text{II.18}$$

With the use of the streamfunction the continuity equation is automatically satisfied. ψ is taken of the form

$$\psi(x,y,t) = \varphi(y)e^{ik(x-ct)} \qquad \text{II.19}$$

Thus separation of variables is used, thereby introducing the eigenvalues k and c. It will be seen that when this type of streamfunction is used in air and water the surface $\eta^{(1)}$ is given by $\eta^{(1)} = \eta_o e^{ik(x-ct)}$.
The wave-nature of the disturbance is clearly exhibited by the x- and t-dependence of the surface. k can be interpreted as the wave-number and c as the phase velocity.

The physical problem throws light on the way to handle the mathematical problem to find η. The physical problem is to find what kind of waves possibly grow due to the instability of the wind (see the introduction). I suppose that the situation is fetch-unlimited (an open sea) and that the wind suddenly rises; subsequently the waves grow. Minimal disturbances are supposed to have been present before the wind rose.

Mathematically the fact that minimal waves are already present is translated by supposing k or c to be given. As growth with time in a uniform sea is considered k is taken to be given and real. c will be determined for each k and may be complex. It is the imaginary part of c that gives the growth:

$$\eta(x,t) = \eta_o e^{(k\mathrm{Im}c)t} e^{ik(x-\mathrm{Re}\,c)t} \qquad \text{II.20}$$

If the problem had been stationary in time and the growth-dependence on fetch had been studied the role of k and c would have been switched (cf. Drazin & Reid, '82 and Kawai, '79).

To find c as a function of k it is necessary to find a streamfunction satisfying the linearized Navier-Stokes equation and the boundary conditions. These equations can be expressed with φ as only first order quantity.

Eliminating the pressure from the Navier-Stokes equations gives the Orr-Sommerfeld equation:

$$(U-c)ik\left(\frac{d^2\varphi}{dy^2}-k^2\varphi\right)-ik\frac{d^2U}{dy^2}\varphi=\nu\left[\frac{d^4\varphi}{dy^4}-2k^2\frac{d^2\varphi}{dy^2}+k^4\varphi\right] \qquad \text{II.21}$$

From the kinematical condition II.13 it follows that a possible solution for the interface is given by

$$\eta^{(1)}=\frac{\varphi_0}{c-U_0}\,e^{ik(x-ct)} \qquad \text{II.22}$$

The suffix $_0$ stands for evaluating the quantity at y = 0. U(0) may be evaluated in air or water, thanks to II.8, $\varphi(0)$ also thanks to II.23. The four other conditions at the interface are (the kinematical condition has been used to determine $\eta^{(1)}$ but the continuity of the normal velocity remains as a condition):

on y = 0 $\quad \varphi_a=\varphi_w \qquad$ II.23

$$\frac{dU_a}{dy}\varphi_a+(c-U_0)\frac{d\varphi_a}{dy}=\frac{dU_w}{dy}\varphi_w+(c-U_0)\frac{d\varphi_w}{dy}$$

$$\frac{\mu_a}{\mu_w}\left[\left(\frac{\frac{d^2U_a}{dy^2}}{c-U_0}+k^2\right)\varphi_a+\frac{d^2\varphi_a}{dy^2}\right]=\left(\frac{\frac{d^2U_w}{dy^2}}{c-U_0}+k^2\right)\varphi_w+\frac{d^2\varphi_w}{dy^2} \qquad \text{II.24}$$

$$\frac{\rho_a}{\rho_w}\left[\left(\frac{dU_a}{dy}-\frac{g}{c-U_0}\right)\varphi_a+(c-U_0+3ik\nu_a)\frac{d\varphi_a}{dy}+\frac{\nu_a}{ik}\frac{d^3\varphi_a}{dy^3}\right]= \qquad \text{II.25}$$

$$=\left(\frac{dU_w}{dy}-\frac{g}{c-U_0}\right)\varphi_w+(c-U_0+3ik\nu_w)\frac{d\varphi_w}{dy}+\frac{\nu_w}{ik}\frac{d^3\varphi_w}{dy^3}-\frac{Tk^2\varphi_w}{\rho_w(c-U_0)} \qquad \text{II.26}$$

at infinite height the boundary conditions become:

$$\varphi_a(\infty)=\left.\frac{d\varphi_a}{dy}\right|_{\infty}=0 \qquad \text{II.27}$$

and at infinite depth:

$$\varphi_w(-\infty)=\left.\frac{d\varphi_w}{dy}\right|_{-\infty}=0 \qquad \text{II.28}$$

The complete set of equations governing waves as perturbations on shear flows is given by II.21-28. Due to the linearization of the equations the boundary conditions are evaluated at y = 0, thus the essentials of a free surface do not enter the problem. The problem can be solved using II.21, 23-28 with a fixed boundary at y = 0. Afterwards with the help of II.22 the actual free surface is found.

Equations II.21, 23-28 form a linear set. When the fourth-order differential equations in air and water have been solved there remain nine unknowns. These are the eigenvalue c and eight constants; the coefficients for the four independent solutions for the streamfunction in air and water each. There are eight boundary conditions: four at the interface and four at infinity. Thus there remains one degree of freedom in the problem.

This situation can be compared to one in quantum mechanics; for instance, with the one-dimensional case of a piecewise constant potential with one potential step (cf. Merzbacher, '70). In the quantum mechanical case there are two boundary conditions at infinity and two at the potential-step. There are five unknowns: the eigenvalue for the energy and four constants; the coefficients for the two independent solutions in both the regions left and right of the potential-step (the Schrödinger equation is a linear second-order partial differential equation). The energy can be solved and the probability function can be known up to a constant. Usually normalization of the probability function is used to remove this degree of freedom.

In the present case we are dealing with a fourth-order differential equation. The essentials are the same, however. The form of the free surface and the eigenvalue for the phase velocity can be found. The streamfunction can be known up to a constant. This implies that only the amplitude of the wave cannot be determined (cf. II.22, where η is already solved in terms of φ and c). Physically this is acceptable as one expects waves with small and large amplitudes to behave the same. Mathematically it can be understood by using $\vartheta = \lambda\varphi$ instead of φ in equations II.21-28, where λ is some constant. All the equations remain exactly the same. This implies that φ can only be known up to a constant. If convenient, a normalization condition can be added to remove this last degree of freedom, as in quantum mechanics.

III. The basic flows.

In the derivation of the Orr-Sommerfeld equation the basic profiles play an important role. These are the wind and the flow in the water when no waves are present.

The interest in the water profile is recent and not much work has been done on it yet. In the present study the flow in the water is considered non-turbulent and described by a profile measured by Kawai ('79). This profile is more or less an exponential one.

The wind speed as function of the height has always attracted much attention. There is experimental evidence for a logarithmic increase of wind speed with height, up to a height of ± 10 m. In the layer very close to the watersurface, which can be regarded as rigid in this context, the air speed increases linearly with height. This layer is called the viscous sublayer and its thickness is typically 0,5 mm.

This air-profile can also be understood theoretically*. I shall give here an argument due to Monin & Yaglom, '71. Turbulence is very important in determining the profile. To describe the turbulence averaging over ensembles is used. All quantities are described by their ensemble-average, indicated by a bar over the symbol, and the deviation from the mean, indicated by a prime. For instance:

$$u = \bar{u} + u'.$$

The momentum flux will be seen to be important. To obtain this flux the equation for conservation of momentum, II.2, is used, but in such a way that turbulence is made explicit. The equation becomes:

$$\rho \frac{\partial \bar{u}_i}{\partial t} + \rho \frac{\partial}{\partial x_\alpha}\left(\bar{u}_i \bar{u}_\alpha + \overline{u_i' u_\alpha'}\right) = \rho \bar{g} - \frac{\partial \bar{p}}{\partial x_i} + \mu \nabla^2 \bar{u}_i \qquad \text{III.1}$$

$$i = 1,2,3$$
$$\alpha = 1,2,3$$

* If the following argument would also be set up to obtain the profile in water the assumption of the rigid surface breaks down.

This is usually called the Reynolds equation. In the derivation the following equality has been used (cf. II.1):

$$u_\alpha \frac{\partial u_i}{\partial x_\alpha} = \frac{\partial}{\partial x_\alpha}(u_i u_\alpha)$$

The Reynolds equation differs from the Navier-Stokes equation by the additional term

$$\tau_{ij} = -\rho \overline{u_i' u_j'}$$

called the Reynolds stress.

In the situation at hand several assumptions can be made. The pressure gradient is taken to be zero, the flow plane-parallel (in the x-direction) and, when averaging over x and t has been performed, the flow is taken to be stationary. Then all quantities depend only on y, the height. The Reynolds equations now take the form (i = x, the other two are identically zero):

$$\nu \frac{d^2\bar{u}}{dy^2} - \frac{d}{dy}\overline{u'v'} = 0 \qquad \text{III.2}$$

the notation is as in chapter II. Equation III.2 states that the flux of x-component of momentum in the vertical direction is constant. By calling this flux τ_0:

$$\tau(y) = \rho\nu\frac{d\bar{u}}{dy} - \rho\overline{u'v'} = \tau_0 \qquad \text{III.3}$$

Exactly at the interface the Reynolds stress has to be zero, as the interface us regarded as rigid. In a thin layer next to the interface the Reynolds stress can be neglected, as it shall be much smaller then the viscous stress. This layer now is called the viscous sublayer. The thickness is $y_1 = \alpha_\nu \nu \sqrt{\frac{\rho}{\tau_0}}$ where α_ν is some constant of the order of unity (cf. Monin & Yaglom, '71, p. 272). In this sublayer equation III.3 reads

$$\rho\nu \frac{d\bar{u}}{dy} = \tau_0 \qquad\qquad y < y_1$$

from which follows directly ($\bar{u}$ must be zero at the interface, cf. II.4)

$$\bar{u}(y) = \frac{u_{*a}^2}{\nu} y \qquad\qquad y < y_1 \qquad\qquad \text{III.4}$$

$$u_{*a}^2 = \frac{\tau_0}{\rho_a} \qquad\qquad \text{III.5}$$

u_{*a} is introduced as a convenient measure for the wind speed, independent of the scaling. The wind speed at the top of the viscous sublayer is $\alpha_\nu u_{*a}$. III.4 states that the flow increases linearly with height in the viscous sublayer.

At heights far above the interface the Reynolds stress can be regarded as much larger than the viscous stress (cf. Monin & Yaglom, '71, p. 272). Then equation III.3 becomes

$$-\rho\overline{u'v'} = \tau_0 \qquad\qquad y > y_0 \gg y_1$$

This equation suggests that the variation of the mean velocity can only depend on τ_0, ρ and y; and certainly not on ν. The mean velocity itself should depend on ν as it is determined by the boundary conditions at y_0, the lower boundary of the domain. At y_0 the velocity is determined by the full equation III.3, so it is also dependent on ν. To return to the velocity gradient, the only combination of τ_0, ρ and y that can be made which has the correct dimension is $\left(\frac{\tau_0}{\rho}\right)^{1/2} \frac{1}{y}$. Thus (cf. III.5)

$$\frac{d\bar{u}}{dy} = A \frac{u_{*a}}{y}$$

This determines the mean flow up to two constants:

$$\bar{u}(y) = A\, u_{*a} \ln y + B \qquad y > y_0 \gg y_1 \qquad \text{III.6}$$

where B may depend on ν . Hereby the logarithmic profile has been made plausible.

In the region between the viscous sublayer and the logarithmic layer the velocity is usually chosen in such a way that the two existing profiles are matched smoothly (as an example, see Kawai '79 or figure 6).

IV. Historical Review.

The two main concurring theories for wave growth are the Phillips' resonance mechanism (see Phillips, '69) and the Miles' mechanism of instability of shear flow. In this paper I work exclusively with the method of instability of shear flow and in this review I shall also confine myself to this mechanism.

This mechanism was introduced for the first time by Jeffreys in 1924, but his theory was too incomplete to be able to agree with experiment. In 1957 Miles tried again, with more success. His central assumption is that at the interface of air and water the pressure induced by the wave motion has a component in phase with the wave height and a component in phase with the wave slope. This can be seen as a combination of the Kelvin-Helmholtz' model, where wave height and pressure are in phase, and the Jeffreys' model, where wave slope is in phase with pressure.

In §V.1. I shall give much attention to the inviscid theory of Miles ('57). I shall give a somewhat different version of the argument Miles used. The alterations are due to Janssen (unpublished notes). My own research (see chapter V) is set up along the same lines as the theory in §IV.1. In §IV.2 the main points of Miles' viscid theory are brought together. In §IV.3 recent numerical and experimental work on the viscous problem is reviewed.

§IV.1 Miles' inviscid theory.

In the air Miles assumes a basic sheared flow U directed in the horizontal plane:

$$y > o: \; U = U(y)$$
$$U(0) = 0$$

y is the vertical coördinate; the water surface is at $y = 0$ when in equilibrium. In the water there is no motion unless waves are present. Miles neglects viscosity and surface tension, which is justified as his main attention is directed at gravity waves ($\lambda > 20$ cm). This leads to the following form of the Orr-Sommerfeld equation, called the Rayleigh equation:

$$(U-c)\varphi'' - \left[k^2(U-c) + U''\right]\varphi = 0 \qquad ' = \frac{d}{dy}$$

As before (chapter II) φ is the height-dependent part of the stream-function and the motion is periodical in time and the horizontal direction. Without viscosity there remain only two boundary conditions at the surface, consistent with the fact that the Rayleigh equation is of second order in the height derivatives. In addition the kinematical boundary condition gives the form of the free surface II.22 (see the discussion at the end of chapter II):

$$\eta = \frac{\varphi_0}{c-U_0}\, e^{ik(x-ct)}$$

A normalization condition is added to the boundary equations; the stream-function in air is taken to be unity at the interface. The remaining boundary conditions plus the normalization equation are:
continuity of normal velocity at the interface:

$$\varphi_a = \varphi_w \qquad \text{on} \quad y = 0$$

the dynamical condition:

$$\frac{\rho_a}{\rho_w}\left[\varphi_a\left(U_a' - \frac{g}{c}\right) + \varphi_a' c\right] = -\frac{g}{c}\,\varphi_w + \varphi_w' c \qquad \text{on} \quad y = 0$$

the normalization:

$$\varphi_a = 1 \qquad \text{on } y = 0$$

the conditions at infinity:

$$\varphi_a = 0 \qquad y \to \infty$$

$$\varphi_w = 0 \qquad y \to -\infty$$

For convenience a translated velocity in the air is introduced:

$$w(y) = c - U(y) \tag{IV.1}$$

and a dimensionless height variable:

$$\xi = ky$$

Then the complete set of equations governing the motion is:

$$y > 0: \quad \varphi_a'' - \left[1 + \frac{w''}{w}\right]\varphi_a = 0 \qquad ' = \frac{d}{d\xi} \tag{IV.2}$$

$$y < 0: \quad \varphi_w'' - \varphi_w = 0 \tag{IV.3}$$

$$y = \infty: \quad \varphi_a = 0 \tag{IV.4}$$

$$y = 0: \quad \varphi_a = 1 \qquad \varphi_w = 1 \tag{IV.5}$$

$$\delta\left[\varphi_a\left(-ww' - \frac{g}{k}\right) + \varphi_a' w^2\right] =$$

$$= -\varphi_w \frac{g}{k} + \varphi_w' w^2 \qquad \delta \equiv \frac{\rho_a}{\rho_w} \tag{IV.6}$$

$$y = -\infty: \quad \varphi_w = 0 \tag{IV.7}$$

For the streamfunction in water there is only one possibility satisfying IV.3, 5 and 7:

$$\varphi_w = e^{\xi} \tag{IV.8}$$

As was shown in chapter II the whole problem concerns the imaginary part of c (or w, as U(y) is real), which determines the growth. To find this growth Janssen introduces the method of expansion of the streamfunction in air and the phase velocity in powers of δ. δ is the air density divided by the water density and is of the order of one-thousands. Miles, though implicitly doing the same is very hazy about this procedure. δ is zero has to be understood as the air molecules having velocities but no mass: and thus no momentum (all velocities remain finite). Thus Janssen sets

$$w(\xi) = w^{(o)}(\xi) + \delta w^{(1)} + \ldots\ldots$$

$$\varphi_a(\xi) = \varphi_a^{(o)}(\xi) + \delta \varphi_a^{(1)}(\xi) + \ldots\ldots \qquad \text{IV.9}$$

In zeroth order the phase speed can be solved from IV.6 + 8 without knowing the streamfunction in air (IV.1 has been used):

$$\left[c^{(o)} \right]^2 = \frac{g}{k} \qquad \text{IV.10}$$

The waves corresponding to this phase velocity are free waves and no growth in time occurs. These free waves are slightly perturbed by the influence of the air-momentum. The dynamical boundary condition in first order in δ reads:

$$2w^{(o)}w^{(1)} = - w^{(o)\prime}w^{(o)} - \frac{g}{k} + \varphi_a^{(o)\prime} w^{(o)2} \qquad \text{IV.11}$$

with the use of IV.5 and 8. It is important to note that, due to the normalization, the expansion of φ_a at the interface becomes trivial: $\varphi_a^{(o)}(0) = 1$ and for $n \neq 0$ $\varphi_a^{(n)}(0) = 0$. IV.11 can be rewritten as

$$w^{(1)} = \tfrac{1}{2} w^{(o)} \left[\varphi_a^{(o)\prime} - 1 - \frac{w^{(o)\prime}}{w^{(o)}} \right] \qquad \text{IV.12}$$

The imaginary part of the phase speed in thus given by (cf. IV.1):

$$\operatorname{Im} c^{(1)} = \tfrac{1}{2} c^{(o)} \operatorname{Im} \varphi_a^{(o)\prime} \qquad \text{IV.13}$$

$U(0) = 0$ has been used.

If the wind profile U is given $w^{(o)}$ is known and $\varphi_a^{(o)}$ can be determined by substituting IV.9 in IV.2. However, from the given expression for the growth it is impossible to see in general what determines whether growth will exist. An expression for Im $\varphi_a^{(o)\prime}$ was found by Miles which does make this possible.

Before handling this I want to show that if Im $\varphi_a^{(o)\prime} = 0$ Miles' assumption of the pressure being neither in phase with the wave height nor the wave slope is justified. The wave induced pressure can be deduced from II.10 by substituting φ, setting $\nu = 0$ and integrating:

$$p_a = \frac{\rho_a}{k} e^{ik(x-ct)} \left[w\varphi_a'' + w'\varphi_a' \right] \qquad ' = \frac{d}{d\zeta}$$

by using II.22 and IV.5 this becomes on the surface:

$$y = 0 : \; p_a = \frac{\rho_a}{k} \left[\varphi_a' + \frac{w'}{w} \right] \eta \qquad \text{IV.14}$$

This implies that φ_a' being complex renders the pressure being out of phase with both the slope and the height of the wave.

Now to return to the question of exactly what determines whether growth exists. First Miles assumed that the wind speed increases monotonically with height. This mean there is one height, called the critical height, where $w = 0$. Then Miles multiplied the zeroth order of IV.2. by $\varphi_a^{(o)*}$, the complex conjugate of $\varphi_a^{(o)}$, and integrated:

$$\int_0^\infty \varphi_a^{(o)*} \varphi_a^{(o)''} d\zeta = \int_0^\infty \left[1 + \frac{w^{(o)''}}{w^{(o)}} \right] . |\varphi_a^{(o)}|^2 \, d\zeta$$

$$\Rightarrow \quad \varphi_a^{(o)*}(0)\, \varphi_a^{(o)'}(0) = - \int_0^\infty |\varphi_a^{(o)'}|^2 + \left[1 + \frac{w^{(o)''}}{w^{(o)}} \right] . |\varphi_a^{(o)}|^2 \, d\zeta \qquad \text{IV.15}$$

The right-hand side of IV.15 can only have an imaginary part due to the singularity of $\frac{1}{w^{(o)}}$ at the critical height. To obtain this contribution contour integration is used. At the critical height the integration path is indented under the real axis to get the right sign for $c^{(1)}$ (cf. Miles, '57); see the picture below.

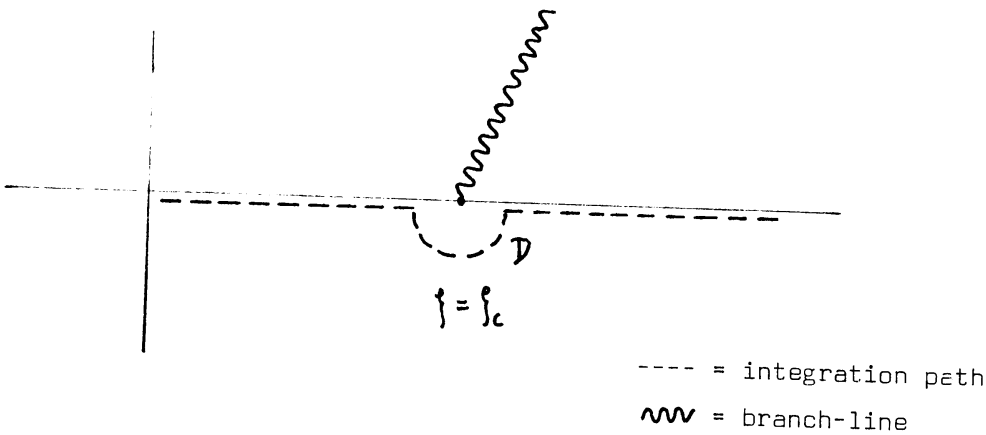

Fig. 1. The integration path of IV.15.

As the integrand is real everywhere on the real axis the only relevant part of the integration path is the half-circle D round ζ_c; the critical height. In a neighborhood of ζ_c $\varphi_a^{(o)}$ can be expanded (Miles, '57):

$$\varphi_a^{(o)}(\zeta) = \varphi_a^{(o)}(\zeta_c)\left[1 + \frac{w^{(o)''}(\zeta_c)}{w^{(o)'}(\zeta_c)}(\zeta-\zeta_c)\ln(\zeta-\zeta_c) + O(\zeta-\zeta_c)\right]$$

By taking the branch-line of the logarithm in the upper half of the complex plane (see picture above) it does not cut the integration patch. The integration over D can be performed and yields (Miles, '57):

$$\operatorname{Im}\,\varphi_a^{(o)'}(0) = \pi\,|\varphi_a^{(o)}(\zeta_c)|^2\,\frac{w^{(o)''}(\zeta_c)}{w^{(o)'}(\zeta_c)} \qquad \text{IV.16}$$

The sought for expression defining the growth can now be given, combining IV.13 and 16:

$$\operatorname{Im}\,c^{(1)} = \tfrac{1}{2}\,c^{(o)}\pi\,|\varphi_a^{(o)}(\zeta_c)|^2\,\frac{w^{(o)''}(\zeta_c)}{w^{(o)'}(\zeta_c)} \qquad \text{IV.17}$$

This is the famous Miles' result which says that the curvature of the wind profile at the critical height determines whether energy-transfer to the waves shall occur.

§IV.2 Miles' viscid theory.

In '59 Miles generalized his theory by admitting viscous terms in his equations. This means he had to use the full Orr-Sommerfeld equation and all four boundary conditions at the interface. Also he dropped the assumption of no flow in the water, though he did assume that the water flow was uniform. Thus the boundary conditions for the basic flows are not satisfied (cf. II.8 ; next to the surface the wind profile is linear, III.4). Miles restricted himself to cases where

$$\frac{u_{*a}^2}{k\nu_a}\,\frac{1}{c} < 10 \qquad \text{IV.18}$$

u_{*a} is given by III.5. In these cases Miles expects the viscous layer above the water surface to be within the laminar sublayer of the undisturbed flow. He comments that as $\frac{u_{*a}^2}{k\nu_a c} \ll R_a^{\frac{1}{2}}$ the inner and outer viscous sublayer will be well separated. The outer viscous sublayer is the layer next to the surface where viscosity is important, the inner is the same near the critical height.

In the air Miles solves the differential equation without specifying the basic profile. He finds two independent solutions for the streamfunction satisfying the boundary conditions at infinity. One is the inviscid solution of his former paper. The other can be found by the method of length-scaling. As scaling parameter Miles uses $R^{\frac{1}{2}}$, where R is the Reynolds number in air. He gives an expression determining the first derivative of the viscid solution:

$$\chi'' - \frac{i(U-c)}{c}\chi = 0 \qquad \text{IV.19}$$

For this solution χ he uses the WKB approximation as he cannot solve it exactly. The solutions still contain the unknown phase speed.

In the water Miles solves the equations and finds two exact solutions. The boundary conditions have to determine the phase speed and the coefficients of the four solutions. As in his former paper Miles first solves for the phase speed by assuming the density in air and the viscosity to vanish. In this way a first approximation to the phase speed is found. This is the same approximation as in the inviscid case: $c_w = \sqrt{\frac{g}{k}}$ for gravity waves. Using this first approximation he solves the coupled equations and finds a second approximation $c_w + c_1$ to c. c_1 is small and given by:

$$c_1 = c_w\left[-2iR_w^{-1} + \tfrac{1}{2}\delta\left(\frac{\varphi_a'}{\varphi_a} + \frac{U_a'}{c_w}\right) - \tfrac{1}{2}\delta\left(1 - \frac{\varphi_a'}{\varphi_a} - \frac{U_a'}{c_w}\right)e^{\frac{1}{4}i\pi}R_a^{-1/2}\right] + \\ + \mathcal{O}\left(R_w^{-3/2}, \delta R_a^{-1/2}R_w^{-1/2}, \frac{\delta u_{*a}^2}{c_w^2}, \delta^2\right) \qquad \text{IV.20}$$

$$R_w = \frac{c_w}{\nu_w k} \qquad\qquad R_a = \frac{c_w}{\nu_a k}$$

The imaginary part of $c = c_w + c_1$ is approximately given by

$$\mathrm{Im}\, c = \tfrac{1}{2}c\delta\, \mathrm{Im}\frac{\varphi_a^{(0)\prime}(0)}{\varphi_a^{(0)}(0)} - 2\frac{g\nu_w}{c_w^2} - \tfrac{1}{2}c_w\delta\left(\frac{g\nu_a}{2c_w^3}\right)^{1/2}$$

Miles interprets the first term as the positive energy transfer from the shear flow to the wave motion, the second as viscous dissipation in water and the third as viscous dissipation in air.

§IV.3 Recent work on the viscous problem.

Valenzuela (1976) also worked on the viscid Orr-Sommerfeld equation to obtain growth-rates but he included the shear flow in the water. He took a linear-logarithmic profile in both air and water. Schematically it looks like figure 2.

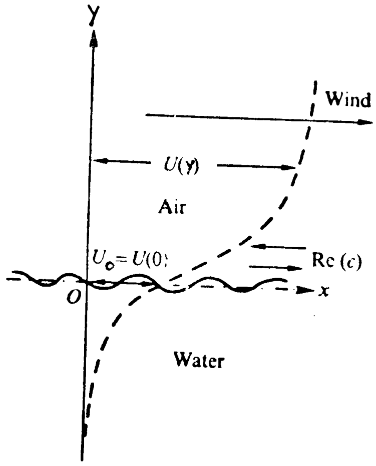

Figure 2. The basic profiles in air and water.
(figure from Valenzuela, '76).

Valenzuela does not attempt an analytical solution: he uses the computer. He finds growth-rates which are in reasonable agreement with the experimental results from Larson & Wright (see Valenzuela, '76). He concludes that the growth of gravity-capillary waves ($\lambda \sim 1$ cm) is due to instability of the shear flows in air and water; and not to the Phillips resonance mechanism. He finds that the shear in the water has considerable influence or the growth rates.

Kawai ('79) made further improvements on the work of Valenzuela. He was especially interested in the initial wavelets, the very first waves to come into existence. He was able to decide between the Phillips' resonce mechanism and the mechanism of instability of shear flow for the initial stage of the generation of waves in favour of the instability mechanism. Valenzuela dealt not with the initial stage but with the subsequent stage of development. Miles himself had taken the resonance mechanism as the one responsible for the initial stage of the generation of waves (see Kawai, '79).Kawai's evidence is numerical as well as experimental.

Kawai solved the viscid equations II.21-28 numerically, same as Valenzuela. The only differences were the relation between u_{*a} and U_r (the wind velocity at a representative height) and the profile they took for the flow in water. Instead of Valenzuela's linear-logarithmic profile Kawai used the one he had measured himself, described by an analytical form by Kurishi (see Kawai, '79). This describes more or less exponential damping.

Kawai's numerical results for the phase velocity show, as can be expected from free-wave theory a minimum value (see figure 3). What is more interesting he also finds, close to this minimum, a maximum for the growth due to the energy input from the wind (again, see figure 3).

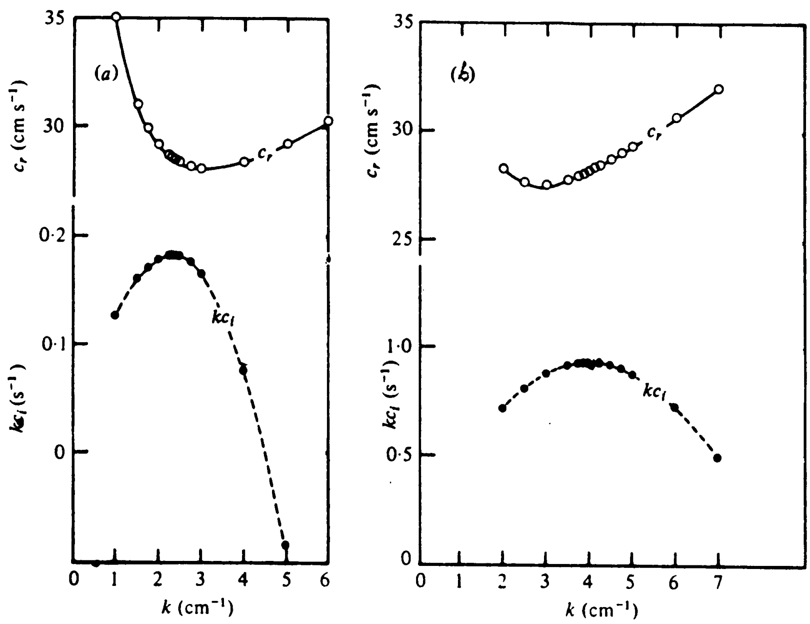

Figure 3. The phase velocity and growth rate as a function of the wavenumber for a) $u_{*a} = 0.136\ ms^{-1}$ and $U_0 = 0.075\ ms^{-1}$; b) $u_{*a} = 0.214\ ms^{-1}$ and $U_0 = 0.098\ ms^{-1}$. Figure from Kawai, '79.

Kawai's measurements show unexpected behavious the very first few seconds after the start of the wind. At first the water surface remains flat, then there appear regular, long-crested wavelets all of the same wavelength. The wavelength and growth of these waves coincide with those of the waves with the maximum growth rate from his numerical work. After another few seconds the waves become irregular and short-crested.

Kawai noted an interesting theoretical feature: in his calculations the critical height is within the linear part of the wind profile. This implies that the curvature at the critical height vanishes and Miles' theory would render the growth zero (cf. IV.17). Some calculations show that the critical height is typically half-way up the viscous sublayer. Kawai does not attempt an explanation of this apparent contradiction.

V The phase velocity and the growth-rate.

Kawai ('79) mentioned the existence of growth of wavelets while the critical height was within the viscous sublayer (see §IV.3). This is in contradiction with the Miles' theory, which predicts growth only when the curvature of the windprofile at the critical height is nonzero. There are three possible explanations of this feature. The first is that Miles' theory is wrong. The second is that the flow in the water, which Miles neglected, causes the growth. The third possibility is that the influence of the viscosity, which Miles also neglected, is such that it causes the instability.

The first explanation might be possible but I do not consider it likely. Miles', derivation of his result (Miles, '57, recapitulated in §IV.1) looks sound, for the case without viscosity or flow in the water.

The second explanation is in general possible. The Miles' inviscid theory about instability of shear flow in air would be just the same for a shear flow in water. The growth, which is proportional to the density, would even be much larger. However, the existence of a critical depth is necessary, which requires the flow at the surface being larger than the phase velocity (supposing the profile exhibits a monotous decrease with depth). For the cases considered by Kawai this condition is not fulfilled (cf. Kawai, '79). Typically, $U_0 = 0{,}06\ ms^{-1}$ and $c = 0{,}28\ ms^{-1}$.

This leaves the third possibility. At first sight it seems unlikely that viscosity causes instability, as it is associated with internal friction which causes energy-loss. However, except dissipating energy viscosity also diffuses momentum. This might have the effect of causing instability. Drazin & Reid ('81) point out that this indeed happens in some cases, notably parallel shear flows.

To be able to check on the third explanation it is necessary to obtain an analytical expression for the growth of gravity-capillary waves in addition to the numerical work done by Valenzuela and Kawai.
Miles ('59) solved for the growth from the inviscid equations but his

interest was different.
He was looking at gravity waves and wanted to show that one might as well use the inviscid equations. Therefore he used basic flows in correspondence with the scale of his waves ($\lambda > 20$ cm); a logarithmic flow in the air and a uniform flow in water.

I have set out to find an expression for the generation of gravity-capillary waves. The method of finding the growth, i.e. of solving the Orr-Sommerfeld equations plus boundary conditions, is the same as in the inviscid case (see §IV.1) The difference in mathematical sense is that now a fourth-order equation has to be solved.

In §V.1 the basic flows in air and water are chosen. In air a linear profile is taken. This results in the linear wind profile model; LWP-model for short. In §V.2 the streamfunction in water is solved. Then in §V.3 general expressions for the phase velocity and growth are found. They are general in the sense that they hold for any windprofile. They are equivalent to IV.10 and 12; the general expressions for the inviscid case. The streamfunction in air is derived in the LWP-model in §V.4. In §V.5 the growth and phase velocity in the LWP-model are found. It is seen that even when the curvature at the critical height vanishes growth of waves is possible. Viscosity causes this instability.

§V.1 The wind and the water flow.

In the main features of the basic flows in air and water I have followed Kawai. This makes comparison of the numerical and analytical results possible.

The windprofile Kawai took is in agreement with the profile derived in chapter III. In the viscous sublayer the profile is given by III.4. As the thickness of the sublayer, y_1, he took $5\frac{\nu_a}{u_{*a}}$.
To match the logarithmic profile at large heights with the linear profile he used the following profile above the viscous sublayer (cf. Kawai, '79) :

$$\left.\begin{aligned} y \geqslant y_1 \; : \; U_a &= 5u_{*a} + U_0 + \frac{u_{*a}}{0{,}4}\,(\alpha - \tanh \tfrac{1}{2}\alpha) \\ \sinh \alpha &= \frac{0{,}8\; u_{*a}}{\nu_a}\,(y - y_1) \end{aligned}\right\} \quad \text{V.1}$$

In his numerical calculations Kawai noticed that the profile influenced the growth up to a height of $\frac{1}{2}\lambda$, λ being the wavelength. This is some 20 times more than the thickness of the viscous sublayer. As an example, take λ = 1,5 cm and y_1 = 0,5 mm (cf. Kawai, '79). This implies that the logarithmic part of the profile also influences the growth.

It is plausible however that the layers closest to the surface dominate the transfers of energy and momentum, which would mean that the influence of the logarithmic profile is but little. Also the important aspect of the wind profile is the linearity up to a height typically twice the critical height. These two considerations led me to take a linear profile in air all the way to infinity. The profile is given by

$$U_a = \frac{u_{*a}^2}{\nu_a}\, y + U_0 \qquad \text{V.2}$$

in agreement with III.4. The model describing the growth of gravity-capillary waves based on this profile is to be called the <u>l</u>inear <u>w</u>ind <u>p</u>rofile model. In §VI.3 I shall consider the validity of the LWP-model.

In water I use instead of Kawai's profile an exponential one. This fits his measurements well at the surface and reasonably well (deviation

$\leqslant$ 15%) down to a depth of 3 mm. This is illustrated in figure 4. The data in the figure were taken at different times but have been normalized to the time of 12 s after the start of the wind. (cf. Kawai, '79). Once an exponential fit is used the profile is determined by the continuity conditions at the interface for the basic flows (II.8) and by the wind profile V.2 :

$$U_w = U_o e^{k_o y} \qquad \text{V.3}$$

$$k_o = \frac{\delta u_{*a}^2}{\nu U_{wo}} \qquad \text{V.4}$$

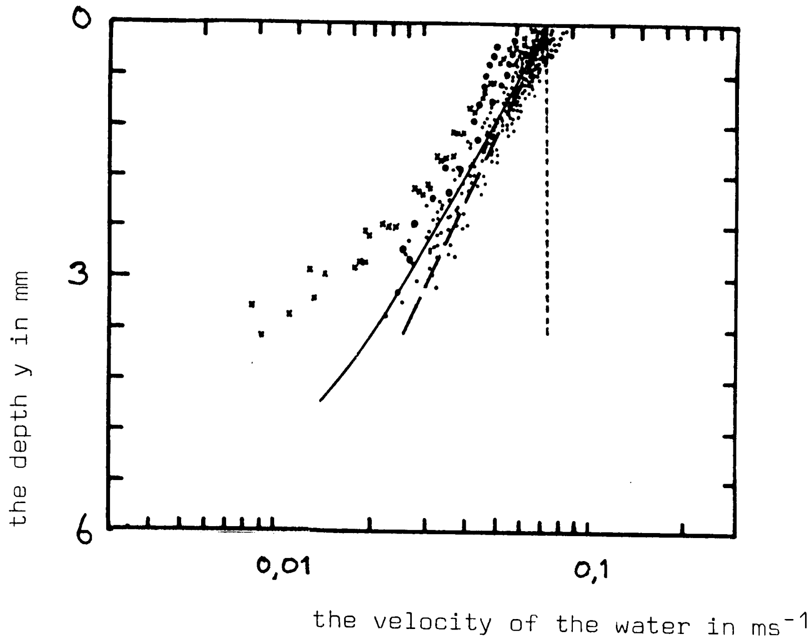

Fig. 4 The flow in the water as a function of depth.
The experimental data are from Kawai, '79.

——— = profile used by Kawai

— — — = exponential profile governed by V.3 and V.4

------ = uniform profile

The profile determined by V.3 and 4 is used only for substitution in the boundary conditions. As I could not solve the Orr-Sommerfeld equation in water using this profile I used a uniform profile, $U_w = U_0$ (see fig. 4). A systematical error is introduced by using different basic profiles for substitution in the boundary conditions and for solving the streamfunction. The functional form of the streamfunction is but an approximation. Errors rise by using this approximation together with the exponential profile. On the other hand errors rise by not using the exponential profile in the boundary conditions. Using the uniform profile makes it impossible to fulfill the continuity conditions for the basic flows (cf. II.8).

A justification is that the flow in the water has much less influence on the growth than the flow in the air. Also the using of the approximated profile has less effect on the Orr-Sommerfeld equation than might be expected. I shall return to this in the next section. In §VI.1 the validity of the approximation of the uniform profile will be considered.

§V.2 The streamfunction in water.

To recapitulate the equations for the streamfunction :

$$\left.\begin{aligned} & w\varphi'' - (w + w'')\varphi = ik\nu_w(\varphi'''' - 2\varphi'' + \varphi) \qquad ' = \frac{d}{d\xi} \\ & \varphi(-\infty) = \varphi'(-\infty) = 0 \\ & \varphi(0) = 1 \end{aligned}\right\} \qquad \text{V.5}$$

Throughout this chapter

$$\xi = ky \qquad \text{V.6}$$

will be used as a dimensionless height variable, negative in water. The problem stated above is a fourth-order differential equation with three boundary conditions given as yet. I shall choose the four independent solutions in such a way that the boundary conditions at infinity rule out two of them. Then the general solution can be written as

$$\varphi_w = A\varphi_1 + B\varphi_2 \qquad \text{V.7}$$

The third boundary condition defines a relation between A and B :

$$1 = A\varphi_1(0) + B\varphi_2(0) \qquad \text{V.8}$$

The value of A (for the moment B is seen as a function of A) and the eigenvalue c, hidden in w, will be determined in §V.3 where the other boundary conditions are used.

Using the form of w to be deduced from IV.1, V.3 and 6 :

$$w = c - U_0 e^{\frac{k_0}{k}\xi} \qquad \text{V.9}$$

makes the finding of an exact solution for the problem stated by V.5 very difficult, if not impossible. Therefore, as mentioned in the foregoing section, w is approximated by a constant :

$$w(\xi) \approx w_o \qquad w_o = c - U_o \qquad \text{V.10}$$

This is allowable when (cf. V.5)

$$\left.\begin{aligned} &\text{for } 0 \geq \xi \geq -3: \quad \left|\frac{w + w'' - w_o}{w_o}\right| \leq 1 \\ &\left|\frac{w - w_o}{w_o}\right| \leq 1 \end{aligned}\right\} \qquad \text{V.11}$$

as $\xi = -3$ coincides with $y = -\frac{1}{2}\lambda$. The conditions V.11 have not been made too severe because the streamfunction has little influence on the growth anyhow. To illustrate that these conditions are usually fulfilled I shall give w and w'' as determined by V.9 at several depth for one case. For the general case I refer to §VI.1.

ξ	y in mm	w in m/s	w'' in m/s	$\left\lvert\frac{w-w_o}{w_o}\right\rvert$	$\left\lvert\frac{w+w''-w_o}{w_o}\right\rvert$
0	0	0.22	0.06	0	0.3
-1	-3	0.26	0.02	0.2	0.3
-3	-8	0.28	0.003	0.3	0.3

Table 1. Comparison of $w = c - U_o e^{\frac{k_o}{k}\xi}$ and w_o for $k_o = k = 360\ m^{-1}$, $U_o = 0.06$ m/s and c= 0.28 m/s

Using V.10 the first of equations V.5 can be written as

$$w_o(\varphi'' - \varphi) = ik\nu\left[(\varphi''-\varphi)'' - (\varphi''-\varphi)\right] \qquad \text{V.12}$$

To find the two independent solutions φ_1 and φ_2 (cf. V.7) satisfying the boundary conditions at infinity I define

$$\chi = \varphi'' - \varphi \qquad \text{V.13}$$

Then V.12 becomes

$$w_o\chi = ik\nu(\chi'' - \chi) \qquad \text{V.14}$$

One solution for χ is

$$\chi_1 \equiv 0$$

This gives a solution to V.12 that also satisfies the inviscid Rayleigh equation; to be called the inviscid solution :

$$\varphi_1 = e^{\xi} \qquad \text{V.15}$$

As the other solution for χ I choose

$$\chi_2 = -iR_w e^{(1-iR_w)^{1/2}\xi} \qquad \text{V.16}$$

Rw, the Reynolds number in water, is defined by

$$R_w = \frac{w_0}{\nu_w k} \qquad \text{V.17}$$

The check on χ_2 being the right choice (I might have taken $\chi = \exp[-(1-iR_w)^{1/2}\xi]$) is that in this way I find a solution φ_2 independent of φ_1 and satisfying the boundary conditions at infinity. The general solution of V.13 can be found by the method of variation of constants. This yields as a solution :

$$\varphi(\xi) = e^{\xi}\int^{\xi} e^{-2\tau}\int^{\tau} e^{\sigma}\,\chi(\sigma)\,d\sigma\,d\tau$$

Substituting V.16 gives

$$\varphi_2(\xi) = e^{(1-iR_w)^{1/2}\xi} \qquad \text{V.18}$$

φ_2 will be called the viscid solution.

The streamfunction is a linear combination of the viscid and the inviscid solution. The proportionality constants are defined by V.8; the normalization at the interface. The streamfunction can be written as

$$\varphi_w(\xi) = \frac{1}{1-A}\left[e^{(1-iR_w)^{1/2}\xi} - Ae^{\xi}\right] \qquad \text{V.19}$$

§V.3 General expressions for the phase velocity and the growth-rate.

Now the streamfunction in water is known and the phase velocity can be approximately solved. Approximately, as an expansion of the continuity equations at the interface in δ (ρ_{air} divided by ρ_{water}) is made (as in §IV.1).
The phase velocity can be found by solving the zeroth order equations; thus the streamfunction and flow in the air have no influence. To obtain the growth-rate a first order expansion in δ is necessary. This implies that the growth-rate does depend on the flow in the air (as is to be expected; the wind causes the growth). In this section the airflow will not be made explicit. In this way a general expression is obtained governing the growth.

The continuity conditions at the interface are given by II.23 - 26. They are here restated with ξ = ky as height variable, supplemented by the normalization condition.

$$
\begin{aligned}
y=0: \quad & \varphi_a = 1 && \text{V.20}\\
& \varphi_w = 1 \\
& -W_a'\varphi_a + W_0\varphi_a' = -W_w'\varphi_w + W_0\varphi_w' \\
& \delta\frac{\nu_a}{\nu_w}\left[\varphi_a(-W_a'' + W_0) + W_0\varphi_a''\right] = \varphi_w(-W_w'' + W_0) + W_0\varphi_w'' \\
& \delta\left[\varphi_a\left(-W_0W_a' - \frac{g}{k}\right) + \varphi_a'W_0^2 + ik\nu_aW_0(3\varphi_a' - \varphi_a''')\right] = \\
& \qquad = \varphi_w\left(-W_0W_w' - \frac{g}{k}\right) + \varphi_w'W_0^2 + ik\nu_wW_0(3\varphi_w' - \varphi_w''') - \frac{Tk\varphi_w}{\rho_w}
\end{aligned}
$$

The wave-dependent quantities are expanded in powers of δ . The basic flows are not expanded as they have nothing to do with the proportionality of ρ_a and ρ_w. The expansion read :

$$
\begin{aligned}
& W_i(\xi) = W_i^{(0)}(\xi) + \delta W_0^{(1)} + \ldots\ldots \qquad i = a \text{ or } w \qquad \text{V.21}\\
& \varphi_i(\xi) = \varphi_i^{(0)}(\xi) + \delta\varphi_i^{(1)}(\xi) + \ldots\ldots
\end{aligned}
$$

Substituting V.21 in V.20 gives the normalization and continuity equations in powers of δ . I shall only need them in zeroth and first order.

Normalization and continuity of normal velocity :

zeroth order:

$$\varphi_a^{(0)} = 1$$

$$\varphi_w^{(0)} = 1 \qquad \text{V.22}$$

first order:

$$\varphi_a^{(1)} = 0$$

$$\varphi_w^{(1)} = 0 \qquad \text{V.23}$$

Conditiors V.22 and 23 will already be used in the following expansions.

Continuity of tangential velccity:

zeroth order:

$$-W_a^{(0)\prime} + W_0^{(0)} \varphi_a^{(0)\prime} = -W_w^{(0)\prime} + W_0^{(0)} \varphi_w^{(0)\prime} \qquad \text{V.24}$$

first order:

$$W_0^{(0)} \varphi_a^{(1)\prime} + W_0^{(1)} \varphi_a^{(0)\prime} = W_0^{(0)} \varphi_w^{(1)\prime} + W_0^{(1)} \varphi_w^{(0)\prime} \qquad \text{V.25}$$

Continuity of shearing stress:

zeroth order:

$$-W_w^{(0)\prime\prime} + W_0^{(0)} + W_0^{(0)} \varphi_w^{(0)\prime\prime} = 0 \qquad \text{V.26}$$

first order:

$$\frac{\nu_a}{\nu_w}\left[-W_a^{(0)\prime\prime} + W_0^{(0)} + W_0^{(0)} \varphi_a^{(0)\prime\prime}\right] = W_0^{(1)} + W_0^{(1)} \varphi_w^{(0)\prime\prime} + W_0^{(0)} \varphi_w^{(1)\prime\prime} \qquad \text{V.27}$$

Continuity of normal stress:

zeroth order:

$$W_0^{(0)2} \varphi_w^{(0)\prime} - W_0^{(0)} W_0^{(0)\prime} + W_0^{(0)} i k \nu_w \left(3\varphi_w^{(0)\prime} - \varphi_w^{(0)\prime\prime\prime}\right) - \frac{g}{k} - \frac{Tk}{\rho_w} = 0 \qquad \text{V.28}$$

first order:

$$2 W_0^{(0)} W_0^{(1)} \varphi_w^{(0)\prime} + W_0^{(0)2} \varphi_w^{(1)\prime} - W_0^{(1)} W_0^{(0)\prime} + W_0^{(1)} i k \nu_w \left(3\varphi_w^{(0)\prime} - \varphi_w^{(0)\prime\prime\prime}\right) +$$

$$+ W_0^{(0)} i \nu_w k \left(3\varphi_w^{(1)\prime} - \varphi_w^{(1)\prime\prime\prime}\right) = W_0^{(0)2} \varphi_a^{(0)\prime} - W_0^{(0)} W_a^{(0)\prime} +$$

$$+ W_0^{(0)} i \nu_a k \left(3\varphi_a^{(0)\prime} - \varphi_a^{(0)\prime\prime\prime}\right) - \frac{g}{k} \qquad \text{V.29}$$

As φ_w is known up to a constant (V.19) :

$$\varphi_w = \frac{1}{1-A}\left[e^{(1-iR_w)^{1/2}\xi} - Ae^{\xi}\right]$$

The expansions of the derivatives of φ_w at the interface can be made explicit (the expansion of φ_w itself is given by V.22 and 23) :

$$\frac{d^n}{d\xi^n}\varphi_w^{(0)}(0) = \frac{(1-iR_w)^{\frac{1}{2}n} - A^{(0)}}{1-A^{(0)}} \qquad \text{V.30}$$

$$\frac{d^n}{d\xi^n}\varphi_w^{(1)}(0) = \frac{1}{1-A^{(0)}}\left[\frac{nc^{(1)}}{2w_0^{(0)}\left(1-\frac{1}{iR_w^{(0)}}\right)} + A^{(1)}\frac{(1-iR_w)^{\frac{1}{2}n}-1}{1-A^{(0)}}\right] \qquad \text{V.31}$$

A formal expansion of A has been introduced, the expansions of c and R_w follow directly from their definitions and V.21.

When no confusion is possible the suffix indicating the evaluation of a quantity in zeroth order will be dropped.

Now the formalism has been developed far enough to enable the phase velocity in zeroth order to be solved. Of the five zeroth order equations at the interface $\varphi_w = 1$ has been used already and here only V.26 and 28, continuity of shearing and normal stress, have to be used. $\varphi_a = 1$ and continuity of tangential velocity are necessary to determine $\varphi_a(\xi)$ and the growth, but not to determine c. They will be used in §V.4 and §V.5.

The unknowns are $A^{(0)}$ and $w^{(0)}$, or $A^{(0)}$ and $c^{(0)}$ as U_0 is considered as known. V.26 gives a relation between them, when V.30 is substituted:

$$-w_w^{(0)''} + w_0^{(0)}\left(1 + \frac{1-iR_w^{(0)}-A^{(0)}}{1-A^{(0)}}\right) = 0 \qquad \text{V.32}$$

V.9 gives w and its derivatives at the interface :

$$w_0 = c - U_0$$

$$n \geqslant 1: \qquad \frac{d^n}{d\xi^n} w_0 = -U_0\left(\frac{k_0}{k}\right)^n \qquad \text{V.33}$$

Substituting this gives :

$$A^{(0)} = 1 - iR_w\left(\frac{w_0}{2w_0 + U_0\left(\frac{k_0}{k}\right)^2}\right) \qquad \text{V.33}$$

V.28 gives another relation between A and w, after substituting V.30 and 32. It reads:

$$w_0^2\,\frac{(1-iR_w)^{1/2} - A}{1-A} + w_0 U_0 \frac{k_0}{k} + w_0 i\nu_w k\,\frac{3(1-iR_w)^{1/2} - (1-iR_w)^{1\frac{1}{2}} + 2A}{1-A} +$$
$$- \frac{g}{k} - \frac{Tk}{\rho_w} = 0 \qquad \text{V.34}$$

To simplify the last equation I expand it in powers of the square root of the Reynolds number, $R_w^{\frac{1}{2}}$. Then I truncate the series. Throughout the following equality is used :

$$n = \tfrac{1}{2}, \tfrac{3}{2}, \ldots \qquad (1-iR_w)^n = (-iR_w)^n\left(1 - \frac{n}{iR_w}\right) + \mathcal{O}\left(R_w^{n-2}\right) \qquad \text{V.35}$$

These expansions and V;33 are substituted in V.34 and the equation is multiplied by 1-A. This yields :

$$w_0^2\left[(-iR_w)^{1/2} - 1 + \frac{iw_0 R_w}{2w_0 + U_0\left(\frac{k_0}{k}\right)^2}\right] - \left(\frac{g}{k} + \frac{Tk}{\rho_w}\right)\frac{iw_0 R_w}{2w_0 + U_0\left(\frac{k_0}{k}\right)^2} +$$
$$+ w_0\left[\frac{U_0 \frac{k_0}{k}\, i\, w_0 R_w}{2w_0 + U_0\left(\frac{k_0}{k}\right)^2} + i\nu_w k\left(-(-iR_w)^{3/2} + \frac{2iw_0 R_w}{2w_0 + U_0\left(\frac{k_0}{k}\right)^2}\right)\right] =$$
$$= 0 + \mathcal{O}\left(R_w^{-1/2}\right) \qquad \text{V.36}$$

Using the definition $R_w = \frac{w_0}{\nu_w k}$ gives the final result for the phase velocity :

$$c^{(0)} = U_0 - \frac{U_0 k_0}{2k} - 2i\nu_w k + \qquad \text{V.37}$$
$$+ \left[\frac{g}{k} + \frac{Tk}{\rho_w} + \left(\frac{U_0 k_0}{2k}\right)^2 + i\nu_w k_0 U_0\left(2 - \frac{k_0}{k}\right)\right]^{1/2} + \mathcal{O}\left(R_w^{-1\frac{1}{2}}\right)$$

Equation V.37 gives the phase velocity in zeroth order as a function of known quantities. When U_0 is small compared to $\left(\frac{g}{k} + \frac{Tk}{\rho_w}\right)^{1/2}$ the last term dominates the expression. In this case the phase velocity is near the free wave velocity. The viscosity gives rise to two imaginary terms. The term $-2i\nu_w k$ represents damping of the wave. The term within the square root represents growth or damping, depending on k. A further discussion of V.37 will be given in chapter VI.

The next step is to obtain an expression for the growth-rate. The growth-rate is given by the imaginary part of $\delta k c$ (see II.20). This implies $c^{(1)}$ has to be solved, and thus the procedure will be exactly the same as when dealing with the phase velocity, except that now first order equations are used. Again I start with the continuity of shearing stress, V.27, combined with V.30 and 31 :

$$\frac{\nu_a}{\nu_w} w_0 (1+\varphi_a'') = c^{(1)}\left[\frac{iR_w+2-2A}{1-A}\right] + \frac{w_0}{1-A}\left[\frac{c^{(1)}}{w_0(1-\frac{1}{iR_w})} + A^{(1)}\frac{-iR_w}{1-A}\right]$$

Using the expression for A V.33 the following can be obtained :

$$A^{(1)} = \frac{-iR_w}{(2w_0 + U_0(\frac{k_0}{k})^2)^2}\left[\frac{\nu_a}{\nu_w}(1+\varphi_a'')w_0^2 + 2\left(w_0 + U_0\left(\frac{k_0}{k}\right)^2\right)c^{(1)}\right] \qquad \text{V.38}$$

The equation is exact and no expansion in powers of R_w. From the equation for continuity of normal stress another relation between $A^{(1)}$ and c can be found, and thus $c^{(1)}$ can be solved. This does demand a lot of computations though. The start is to substitute V.30 and 31 in V.29 and multiply the equation by $\frac{1-A}{w_0}$:

$$2c^{(1)}\left[(1-iR_w)^{1/2} - A\right] + \frac{c^{(1)}U_0(1-A)}{w_0} + \frac{c^{(1)}i\nu_w k}{w_0}\left[-(1-iR_w)^{3/2} + \right.$$

$$\left. + 3(1-iR_w)^{1/2} - 2A\right] + w_0\left[\frac{c^{(1)}}{2w_0(1-\frac{1}{iR_w})} + A^{(1)}\frac{(1-iR_w)^{1/2}-1}{1-A}\right] +$$

$$+ i\nu_w k\left[\frac{-3c^{(1)}}{2w_0(1-\frac{1}{iR_w})} + \frac{3c^{(1)}}{2w_0(1-\frac{1}{iR_w})} + \frac{A^{(1)}}{1-A}\left(-(1-iR_w)^{3/2} + 3(1-iR_w)^{1/2} - 2\right)\right] =$$

$$= (1-A)w_0\varphi_a' - \frac{(1-A)g}{w_0 k} + (1-A)\left[U_a' + i\nu_a k(3\varphi_a' - \varphi_a''')\right]$$

This equation is expanded in powers of $R_w^{\frac{1}{2}}$ and the relation between $A^{(1)}$ and $c^{(1)}$, V.38, is used together with the definition of R_w. The equation will be truncated after fewer orders of $R_w^{\frac{1}{2}}$ than in the former case as that was a zeroth order equation in δ and this one is first order. The orders of U_a', φ_a', φ_a'' and φ_a''' are not known so these terms will be kept, though the factors multiplying them will be truncated. It is very important to keep the derivatives of the streamfunction in the air as they, if anything, will cause the growth. After some computations the equation becomes surprisingly simple :

$$c^{(1)} i R_w \frac{2w_0 + U_0 \frac{k_0}{k}}{2w_0 + U_0 (\frac{k_0}{k})^2} = i R_w \frac{w_0}{2w_0 + U_0 (\frac{k_0}{k})^2} \Big[U_a' + $$
$$+ i\nu_a k (3\varphi_a' - \varphi_a''') \Big] + \frac{\varphi_a'' w_0^2 \frac{\nu_a}{\nu_w}}{2w_0 + U_0 (\frac{k_0}{k})^2} \left(-1 + (-i R_w)^{-1/2} \right) +$$
$$+ \frac{\varphi_a' w_0^2 i R_w}{2w_0 + U_0 (\frac{k_0}{k})^2} - \frac{i R_w g}{k (2w_0 + U_0 (\frac{k_0}{k})^2)} + \mathcal{O}(R_w^0)$$

In air I also define a Reynolds number :

$$R_{ao} = \frac{w_0}{\nu_a k} \qquad \text{V.39}$$

The expression for $c^{(1)}$ then reads :

$$c^{(1)} = \Big[\varphi_a' w_0 + U_a' - \frac{g}{k w_0} \Big] \frac{w_0}{2w_0 + U_0 \frac{k_0}{k}} +$$
$$+ \frac{i}{R_{ao}} \frac{w_0^2}{2w_0 + U_0 \frac{k_0}{k}} \Big[3\varphi_a' + \varphi_a'' (1 - (-i R_w)^{-1/2}) - \varphi_a''' \Big] + \mathcal{O}(R_w^{-1}) \qquad \text{V.40}$$

All quantities are to be evaluated at y = o

The growth rate β for the amplitude of the wave is obtained by adding the imaginary parts of V.37 and δ times V.40 and multiplying by k; approximately this becomes :

$$\beta = -2 i \nu_w k^2 + k \,\mathrm{Im} \Big[\frac{g}{k} + \frac{Tk}{\rho_w} + \left(\frac{U_0 k_0}{2k}\right)^2 + i \nu_w k_0 U_0 (2 - \frac{k_0}{k}) \Big]^{1/2} +$$
$$+ \frac{k \delta w_0^2}{2w_0 + U_0 \frac{k_0}{k}} \Big[\mathrm{Im}\, \varphi_a' + \frac{1}{R_{ao}} \mathrm{Re} (3\varphi_a' + \varphi_a'' - \varphi_a''') \Big] \qquad \text{V.41}$$

The real part of V.40 gives a correction on the phase velocity (see §VI.4), but now there are additional terms due to viscosity proportional to R_{ao} or even larger.These terms can have a significant contribution to the growth rate, as can be seen in V.41.

§V.4 The streamfunction in air.

The streamfunction in air is solved for the special case of the linear wind profile model. This profile was discussed in §V.1. Let V be given by

$$V = \frac{u_{*a}^2}{\nu_a k} \qquad \text{V.42}$$

then $U_a = V\xi + U_o$. The Orr-Sommerfeld equation plus three of the boundary conditions in the LWP-model read (ξ = ky as variable) :

$$(c - U_o - V\xi)(\varphi'' - \varphi) = ik\nu_a\left[(\varphi'' - \varphi)'' - (\varphi'' - \varphi)\right]$$

$$\varphi(\infty) = \varphi'(\infty) = 0 \qquad \text{V.43}$$

$$\varphi(0) = 1$$

To simplify the differential equation I introduce several new quantities. First the height coördinate is translated :

$$\eta - \eta_o = \xi$$

$$\eta_o = -\frac{1}{V}(c - U_o) \qquad \text{V.44}$$

The translation is such that $\eta = \eta_o$ is at the surface and $\eta = 0$ is at the critical height. Then another Reynolds number is introduced :

$$R_a = \frac{u_{*a}}{\nu_a k} \qquad \text{V.45}$$

Typically, $u_{*a} = 0.14$ m/s and, for gravity-capillary waves, $w_o = 0.22$ m/s (see §VI.1). Thus R_a and $R_{ao} = \frac{w_o}{\nu_a k}$ are of the same order. Finally I define

$$\varepsilon = (iR_a^2)^{-1/3} \qquad \text{ph } \varepsilon = -\frac{\pi}{6} \qquad \text{V.46}$$

In terms of $\chi = \varphi'' - \varphi$ (as introduced in V.13) and with the substitution of the newly defined quantities the Orr-Sommerfeld equation reads :

$$\eta \chi = \varepsilon^3 (\chi'' - \chi) \qquad \text{V.47}$$

An analoque of the inviscid solution that was found in water satifies V.47 :

$$\chi_1 = 0$$

$$\Rightarrow \quad \varphi_1 = e^{-\eta} \qquad \text{V.48}$$

This solution will again be indicated as the inviscid solution.

More solutions of V.47 are given by (cf. Drazin & Reid, '82) :

$$\chi(\eta) = A_n(\zeta + \varepsilon^2)$$

The functions $A_n(z)$ are the generalized Airy functions. They are given by (Drazin & Reid, '82) :

$$A_n(z) = e^{\frac{n2\pi i}{3}} \, Ai(z\, e^{\frac{n2\pi i}{3}}) \qquad n = 1, 2, 3$$

The viscid solution φ_2 is obtained using the relation $\chi = \varphi'' - \varphi$.
If $\chi_2 = -Ai(\frac{\eta}{\varepsilon} + \varepsilon^2)$ is chosen the boundary conditions at infinity are satisfied. The viscid solution then is :

$$\varphi_2(\eta) = \int_{\eta}^{\infty} \sinh(\eta - \eta')\, Ai(\zeta' + \varepsilon^2)\, d\eta' \qquad \text{V.49}$$

$$\zeta = \frac{\eta}{\varepsilon} \qquad \text{V.50}$$

The streamfunction in air is a linear combination of the viscid and the inviscid solutions. Using the normalization at the interface it can be written as :

$$\varphi_a = \alpha e^{-\eta} + \frac{1 - \alpha e^{-\eta_0}}{\varphi_2(\eta_0)} \varphi_2(\eta) \qquad \text{V.51}$$

The streamfunction in the LWP-model is given explicitly by V.51.

There rises a difficulty however when one wants to substitute φ_a in the general expression for the growth, V.41. The values of $\varphi_2(\eta_0)$ and its derivatives will have to be approximated. As a first, rough approximation I take

$$\eta_0 = 0 + \mathcal{O}(R_a^{-1})$$

and use the follwing expansion of V.49 for $\eta = 0$ (Reid, '74) :

$$\int_\infty^0 f(\eta')Ai(\zeta'+\varepsilon^2)\,d\eta' = \varepsilon f(0)\int_\infty^0 Ai(z)dz + \varepsilon^2 f'(0)Ai'(0) + \mathcal{O}(\varepsilon^3) \qquad \text{V.52}$$

The viscid solution at the interface becomes in this way :

$$\varphi_2(\eta_0) = -\varepsilon^2 Ai'(0) + \mathcal{O}(R_a^{-1}) \qquad \text{V.53}$$

The derivatives of φ_2 also have to be approximated. Differentiating V.49 gives

$$\varphi_2'(\eta) = \int_\eta^\infty \cosh(\eta-\eta')\,Ai(\zeta'+\varepsilon^2)\,d\eta'$$

Using the same approximations as for φ_2 it follows that at the interface ($\int_\infty^0 Ai(z)\,dz = -\frac{1}{3}$, Abramowitz & Stegun, '65 p 478)

$$\varphi_2'(\eta_0) = -\tfrac{1}{3}\varepsilon + \mathcal{O}(R_a^{-1}) \qquad \text{V.54}$$

The second derivative is best obtained from the relation $\chi_2 = \varphi_2'' - \varphi_2$:

$$\varphi_2''(\eta) = -Ai(\zeta+\varepsilon^2) + \varphi_2(\eta)$$

Approximately this becomes at the interface

$$\varphi_2''(\eta_0) = -Ai(0) + \mathcal{O}(R_a^{-1/3}) \qquad \text{V.55}$$

From the second derivative the third is obtained.
At the interface it is given by :

$$\varphi_2'''(\eta_0) = -\frac{Ai'(0)}{\varepsilon} + \mathcal{O}(R_a^0) \qquad \text{V.56}$$

The effect that the approximations, made inside the LWP-model for the streamfunction and its derivatives at the interface, have on the growth-rate will be considered in §VI.2.

§V.5 The growth-rate in the linear wind profile model.

In the foregoing section the streamfunction in air has been solved for a linear wind profile. The linear wind profile approximates the true profile for gravity-capillary waves, as was shown in §V.1. Combining this profile with the general results of §V.3 gives a theoretical explanation for the experimental results by Kawai concerning the growth of gravity-capillary waves.

The phase velocity in zeroth order as described by V.37 is not dependent on the airflow. No new contributions are to be made here to this description The expressions for the first order corrections in δ to c and for the growth-rate (V.40 and 41) do contain derivatives of the streamfunction in air. These derivatives will be calculated for the LWP-model in the following paragraphs.

The streamfunction in air is given by V.51 :

$$\varphi_a(\eta) = \alpha e^{-\eta} + \frac{1-\alpha e^{-\eta_0}}{\varphi_2(\eta_0)}\,\varphi_2(\eta)$$

I shall take the zeroth order expansion of this expression by substituting $\alpha^{(0)}$ for α and $\eta_0^{(0)}$ for η_0 .
The suffixes $^{(0)}$ will be dropped. Of the five zeroth order normalization plus continuity conditions at the interface one has not been used yet, notably the continuity of tangential velocity (V.24). This equation is now used to calculate α . Besides V.24 I use V.2, 30, 42, and 51. This gives :

$$V + w_0\left(-\alpha e^{-\eta_0} + (1-\alpha e^{-\eta_0})\frac{\varphi_2'}{\varphi_2}\right) = U_0\frac{k_0}{k} + w_0\,\frac{(1-iR_w)^{1/2} - A}{1-A}$$

In this expression as in the follwing φ_2^n stands for $\varphi_2^n(\eta_0)$.
The expression can be approximated by setting $\eta_0 = 0 + \mathcal{O}(R_a^{-1})$ (see §V.4) and $A = -iR_w\,\frac{w_0}{w_0 2 + U_0(\frac{k_0}{k})^2} + \mathcal{O}(R_w^0)$ (cf. V.33). Then α is given by :

$$\alpha = \frac{\frac{V}{w_0} - \frac{U_0 k_0}{2k} - 1 + \frac{\varphi_2'}{\varphi_2}}{1 + \frac{\varphi_2'}{\varphi_2}} + \mathcal{O}\left(\alpha R_a^{-1},\ \left(1+\frac{\varphi_2'}{\varphi_2}\right)^{-1} R_w^{-1/2}\right)$$

Using the approximations for φ_2 and φ_2' V.53 and 54 this becomes :

$$\alpha = \frac{3V\varphi_2}{W_0\,\varepsilon} + \mathcal{O}(R_a^{-1}) \qquad \text{V.57}$$

It will turn out best not to substitute $\varphi_2(\eta_0)$; the order of magnitude of $\varphi_2(\eta_0)$ has been used though.

It may be noticed that the calculations are much less exact in this section than they were in §V.3, describing the general growth. This is partly form necessity and partly because in general it is not known which terms are important, while in this model it is possible to see which terms dominate the equations.

Now α is known and φ_a', φ_a'' and φ_a''' can be determined. To keep the errors small I calculate φ_a' directly from V.24; this gives :

$$\varphi_a' = -\frac{V}{W_0} + \mathcal{O}(R_w^0) \qquad \text{V.58}$$

φ_a'' and φ_a''' follow with the use of V.51 and of the approximations for φ_2'' and φ_2''' V.55 and 56.

$$\varphi_a'' = \alpha - \frac{\alpha}{\varphi_2}\varphi_2'' + \mathcal{O}\left(R_a^{-1}\,\alpha\left(1 - \frac{\varphi_2''}{\varphi_2}\right)\right)$$

$$\Rightarrow \quad \varphi_a'' = \frac{3V\,Ai(0)}{W_0\,\varepsilon} + \mathcal{O}(R_a^{4/3}) \qquad \text{V.59}$$

and

$$\varphi_a''' = -\alpha - \frac{\alpha}{\varphi_2}\varphi_2''' + \mathcal{O}\left(R_a^{-1}\,\alpha\left(1 - \frac{\varphi_2'''}{\varphi_2}\right)\right)$$

Using the fact that $Ai'(0)$ is negative (Abramowitz & Stegun, '65, p 446) and V.56 this becomes :

$$\varphi_a''' = -\frac{3V\,|Ai'(0)|}{W_0\,\varepsilon^2} + \mathcal{O}(R_a^{5/3}) \qquad \text{V.60}$$

These expressions for φ_a', φ_a'' and φ_a''' are substituted into the first order correction to c V.40 :

$$c^{(1)} = \frac{i\,3\,W_0\,V\,|Ai'(0)|}{(2W_0 + U_0\frac{k_0}{k})\,\varepsilon^2 R_{a0}} + \mathcal{O}(R_a^{2/3}) \qquad \text{V.61}$$

In the LWP-model the growth β of the aplitude of a wave can be approximated by substituting V.61 in V.41. It has to be kept in mind that this is but a crude estimate of the growth. This is not due to limitations of the LWP-model but to the rough approximations made for the values of the streamfunction and its derivatives at the interface. This subject will be futher treated in §VI.2.

With the use of V.61, $ph\ \varepsilon^{-2} = \frac{1}{3}\pi$ (cf. V.46) and an approximation to the imaginary part of V.37 the growth becomes :

$$\beta = -2\nu_w k^2 + \frac{k\,\nu_w U_0 k_0 (2-\frac{k_0}{k})}{2W_0 + \frac{U_0 k_0}{k}} + \qquad \text{V.62}$$
$$+ \frac{\delta\,3\,k\,V\,|Ai'(0)|\,u_{*a} R_a^{1/3}}{2\,(2W_0 + \frac{U_0 k_0}{k})} + \mathcal{O}(R_a^{2/3})$$

It will be shown in the calculations in the next section that for certain wavenumbers V.62 describes positive growth. Looking again at the general expression it is seen that the growth is caused by a term proportional to ν_w and by a term proportional to $\nu_a \varphi_a'''$. This makes it plausible that viscosity itself is the cause of the instability.

VI. Discussion.

In chapter V the phase velocity and the growth-rate have been determined subject to certain assumptions. In this chapter the values for these quantities calculated in this way will be compared to numerical values by Kawai (§VI.1 and §VI.2). In these sections the range of validity of the analytical expressions is also derived.
Next the expression for the growth derived in this study will be compared with the expression Miles derived in his viscid theory (§VI.3).
In §VI.4 the limit will be taken to very large Reynolds numbers, which makes comparison with Miles' inviscid theory possible.
In §VI.5 attention will be paid to the possibility of growth while the density in the air is zero. Finally in §VI.6 a new aspect of the theory will be introduced. The conservation of energy will be considered and described mathematically. This will lead to an answer to the question: "Which contributes most to the energy transfer from wind to waves; the normal pressure or the shearing stress ?"

§VI.1 The phase velocity.

Equation V.37 is an expression for the phase velocity.
This expression has been derived subject to the assumption that in the Orr-Sommerfeld equation a constant waterflow could be used (see §V.2). Actually V.37 gives only the zercth order expansion of the phase velocity; the real part of V.61 geives a first order correction in δ to it. However, as this correction is 1% or less of the zeroth order expansion it can be dropped. This implies that the approximation of the wind profile has no influence on the values of the phase velocity.

To be able to compute phase velocities from V.37 U_o and k_o have to be known. The product $U_o k_o$ is a function of the friction velocity (V.4) :

$$U_o k_o = \frac{\delta u_{*a}^2}{\nu_w}$$

Also there is a value for U_o corresponding to each wind speed. This was shown experimentally by Kawai, '79. I use his relation between U_o and u_{*a}. These two relations determine U_o and k_o as a function of u_{*a}, as can be seen in table 2.

u_{*a} in m/s	U_o in m/s	k_o in m/s
0.136	0.075	250
0.170	0.096	300
0.214	0.098	470
0.248	0.102	600

table 2. U_o and k_o as a function of u_{*a}.
Based on V.4 and measurements by Kawai, '79.

In fig. 5 the phase velocity is shown as calculated from V.37 and table 2. For comparison the values for the phase velocity obtained numerically by Kawai, '79 are also shown in fig. 5. As discussed in §V.1 I have used the same basic flows as Kawai. Only I was forced to make approximations to be able to solve the equations analytically. Kawai used the exact flows. This makes a check on the validity of the anylytical expressions derived in the present study possible.

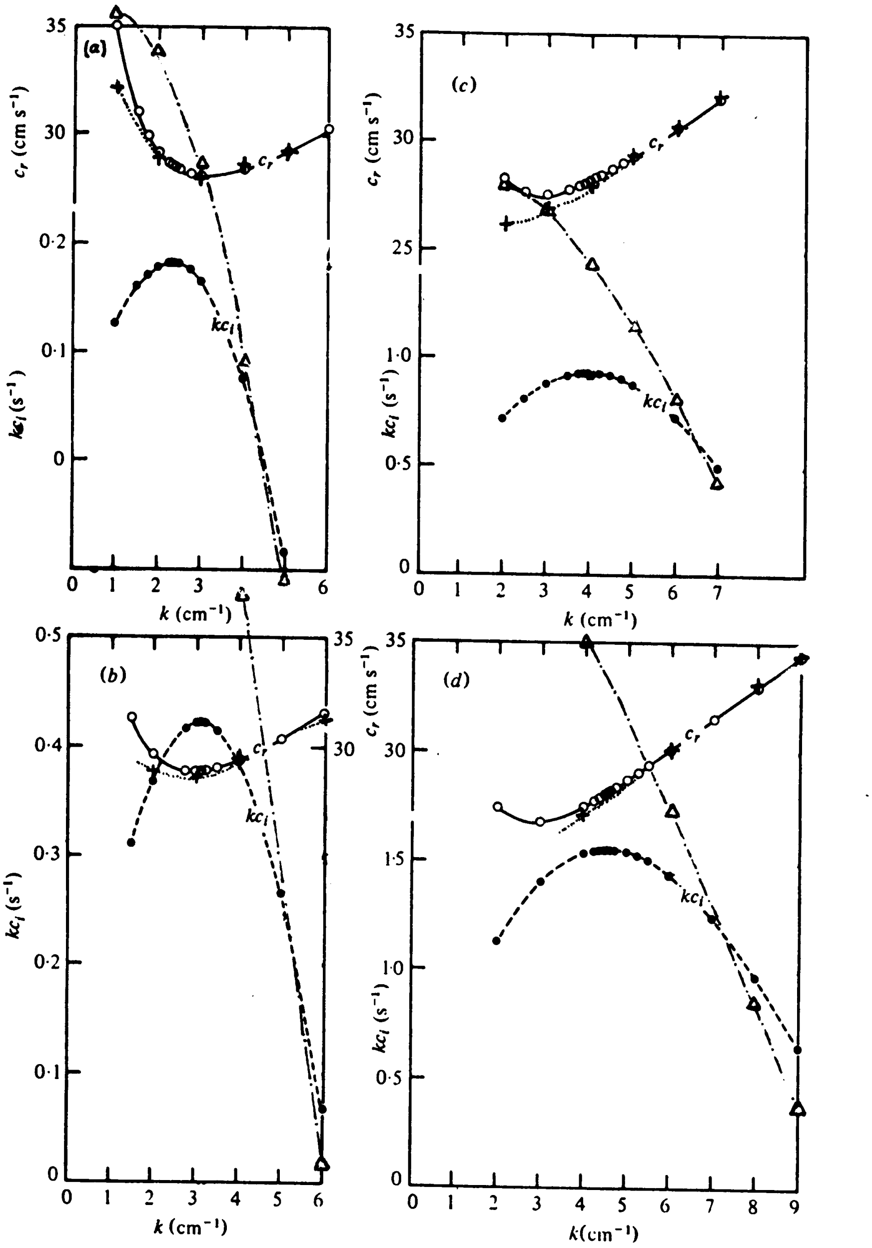

Fig. 5. The phase velocity and the growth rate as a function of the wavenumber for several wind speeds.

a) u_{*a} = 0.136 m/s

b) u_{*a} = 0.170 m/s

c) u_{*a} = 0.214 m/s

d) u_{*a} = 0.248 m/s

o———o = the phase velocity as found by Kawai, '79.

+··········+ = the phase velocity as given by V.37 and table 2.

•– – –• = the growth-rate as found by Kawai, '79.

Δ–·–·–Δ = the growth-rate as given by V.62.

It is seen in fig. 5 that for all wind speeds considered the analytical values for the phase velocity agree with the numerical values. For the smaller wavenumbers a deviation is shown; for the larger wavenumbers the agreement is very good.

The deviation at the smaller wavenumbers can be understood by looking at the range of validity of V.37. V.37 is valid only when the approximation of the uniform profile in water is justified. Necessary conditions for this approximation are (cf. V.11) :

$$\text{for} \quad -3 \leq \zeta \leq 0 \qquad \left| \frac{W + W'' - W_0}{W_0} \right| \leq 1$$

$$\left| \frac{W - W_0}{W_0} \right| \leq 1 \qquad \text{VI.1}$$

where w is given by V.9 :

$$w = c - U_0 e^{\frac{k_0}{k}\zeta} \qquad w_0 = c - U_0$$

The second condition of VI.1 is fulfilled only if

$$U_0 \leq \tfrac{1}{2}c \qquad \text{VI.2}$$

as

$$\forall\, k, \zeta \qquad \left| \frac{W - W_0}{W_0} \right| \leq \left| \frac{U_0}{c - U_0} \right|$$

The first expression of VI.1 is, for small wavenumbers, largest for $\zeta = 0$. Then:

$$\zeta = 0 : \quad \left| \frac{W + W'' - W_0}{W_0} \right| = \frac{\left(\frac{k_0}{k}\right)^2 U_0}{c - U_0} \qquad \text{VI.3}$$

Combining VI.1,2,3 and V.4 gives the minimum wavenumber for equation V.37 to be valid:

$$k_{min} = \max\left(\tfrac{1}{2}c \,,\ \frac{\delta u_{*a}^2}{\nu_w \left(U_0 (c - U_0)\right)^{1/2}} \right) \qquad \text{VI.4}$$

These minimum values are given in table 3 for the values of the friction velocity used in fig. 5.

u_{*a} in m/s	0.136	0.170	0.214	0.248
k_{min} in m^{-1}	150	210	350	460

table 3. The minimum wavenumber for V.37 to be valid, as given by VI.4, for several wind speeds.

It is seen that for each wind speed the analytical values for the phase velocity agree well with the numerical values within the estimated range of validity of equation V.37.
It may further be noted that when $U_0 = 0$ the range of validity is infinite.

§VI.2 The growth-rate.

Equation V.41 gives a general expression for the growth-rate, valid for any wind profile. In the case of a linear wind profile this equation can be approximated by V.62. To compute actual values from V.62 w_0, U_0 and k_0 have to be known.
w_0 stands for the zeroth order expansion of $w_0 = c - U_0$ and is given by V.37. U_0 and k_0 are given in table 2 for several friction-velocities.
In fig. 5 the growth-rates are plotted as a function of the wave-number for several wind speeds.
The numerical values obtained by Kawai (see §VI.1) are shown in the same figure. The wavenumber of the maximum of the curve Kawai computed coincides with the wavenumber of the first waves that are generated in experimental situations (see §IV.3).

The most important conclusion to be drawn from fig. 5 is that positive growth-rates are found. This is in contrast with Miles' inviscid theory but in agreement with numerical calculations. This testifies to the hypothesis that viscosity is essential for the growth of gravity-capillary waves.

The growth-rates in the LWP-model are of the same order of magnitude as the numerical growth-rates. In all four cases considered in fig. 5 there even is good agreement between LWP- and numerical values for Reynolds numbers close to 23. These characteristics can be understood by a consideration of the range of validity of eq. V.62.

V.62 is a special case of V.41. To obtain V.41 the approximation of the uniform profile in water was made, this was discussed in §VI.1. The theoretical estimate of the range of validity of this approximation is $k \gg k_{min}$; $k_{min} = \max\left(\tfrac{1}{2}c, \dfrac{\delta u_{*a}^2}{\nu_w\left[U_0(c-U_0)\right]^{1/2}}\right)$ (VI.4).

V.41 is thus valid for $k \gg k_{min}$ and this implies that V.62 can only be valid for $k \gg k_{min}$. But to obtain V.62 from V.41 the LWP-model was used. This makes a further restriction on the range of validity of V.62.

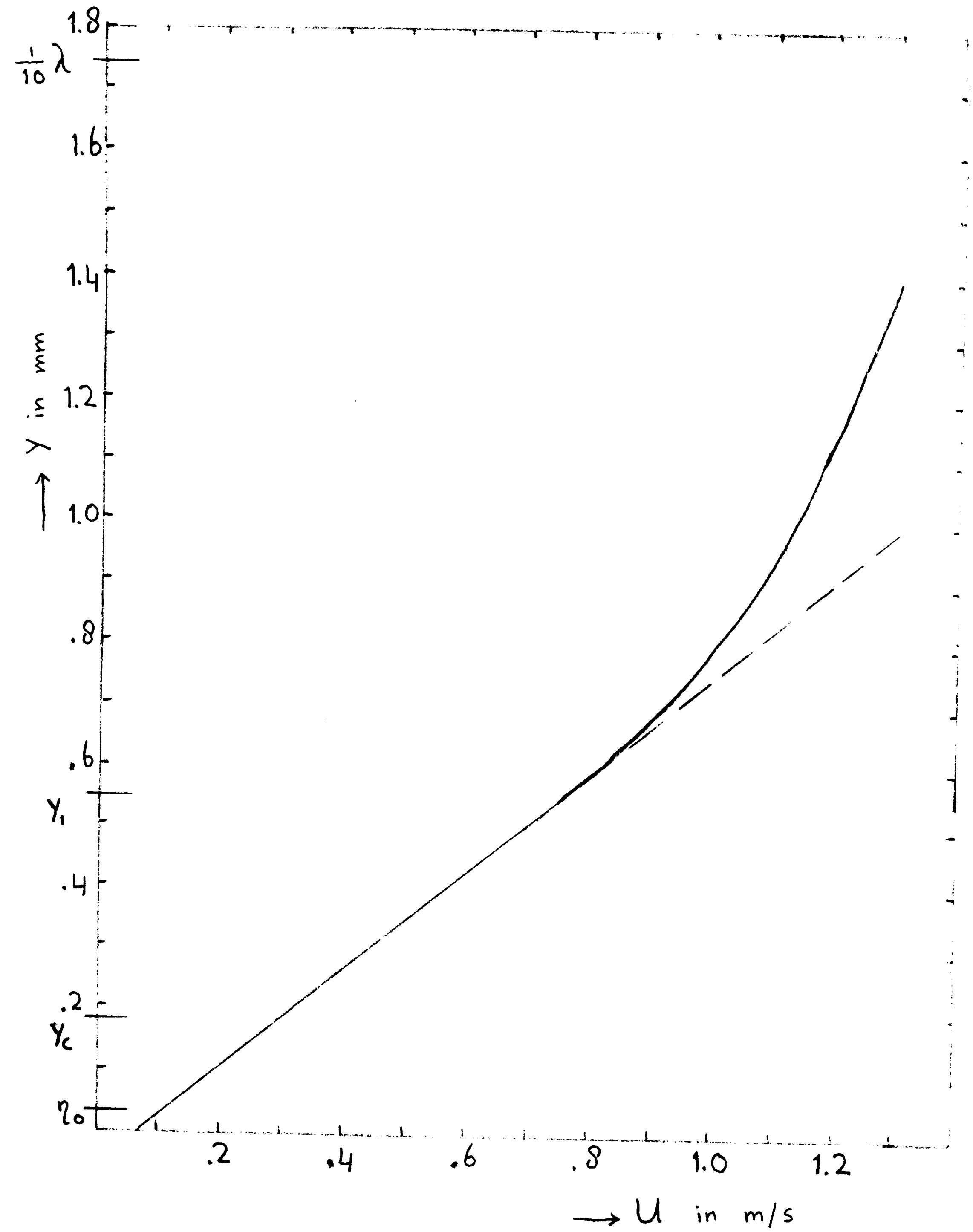

fig. 6 The exact profile and the linear wind profile.

—— = exact profile, – – – = LWP, η_0= the amplitude of the wave, y_c = the critical height, $\frac{1}{10}\lambda$ = one tenth of the wavelength (η_0, y_c and λ for a typical wave (Kawai, '79)), y_1 = the thickness of the viscous sublayer, here $y_1 = 5\frac{\nu_a}{u_{*a}}$.

To obtain this range of validity first the height where the exact profile (V.1 and 2) and the linear profile (V.2) start to deviate considerably is estimated; see fig. 6. This height of course depends on y_1, the thickness of the viscous sublayer. In this estimation I shall use $y_1 = 5 \frac{\nu_a}{u_{*a}}$ as Kawai's calculations, drawn in fig. 5, are based on this value for y_1. It is seen in fig. 6 that for $y \leqslant 8 \frac{\nu_a}{u_{*a}}$ ($y \leqslant 0.88$ mm in fig. 6) the deviation of the two profiles is less than 10%.

The next step is to notice that φ_2''', the viscous part of φ_a''', determines the growth in the LWP-model (see V.60 and the remark at the end of §V.5). φ_2''' is given by (cf. V. 55):

$$\varphi_2'''(\eta) = -\frac{Ai'(\zeta + \varepsilon^2)}{\varepsilon} + \int_\eta^\infty \cosh(\eta - \eta') Ai(\zeta' + \varepsilon^2) d\eta' \qquad \text{VI.5}$$

Thus its characteristics are determined by the Airy-function. For real arguments the Airy function as well as its derivative decrease rapidly and monotonously and when this argument equals three their values are less than 3% of their values at zero, see fig. 7.

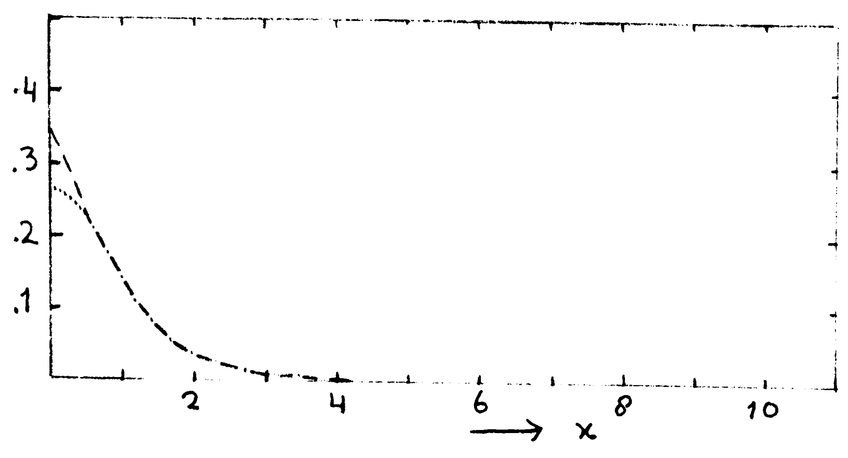

fig. 7. The Airy-function (figure from Abramowitz & Stegun, '65).
---- = Ai (x), ... = -Ai'(x).

I assume that the Airy-function dominates the behaviour of the integrand (which is true at infinity) and I skip the fact that the arguments in VI.5 have a phase of $\frac{1}{6}\pi$ instead of being real. Then it follows, the arguments being more or less $\frac{\eta}{\varepsilon}$, that for

$$\frac{\eta}{|\varepsilon|} \geqslant 3 \qquad \text{VI.6}$$

it does not matter much whether you take the exact profile or the linear one. In other words, I assume the profile for $\frac{\eta}{|\varepsilon|} \geqslant 3$ to have little influence on the streamfunction for $\frac{\eta}{|\varepsilon|} \leqslant 3$ and I assume that the streamfunction for $\frac{\eta}{|\varepsilon|} \geqslant 3$ hardly contributes to the total value of the integral in VI.5. The value of 3 is somewhat arbitrary, it might as well have been $2\frac{1}{2}$ or $3\frac{1}{2}$.

This suggests that as the boundary of the domain of validity for the LWP-model that wavenumber is taken for which the height $\eta = 3|\varepsilon|$ coincides with $y = \frac{8\nu_a}{u_{*a}}$, the height where the LWP and the exact profile start to deviate substantially. If this definition is accepted it follows that the minimum wavenumber is ($\eta = ky + \mathcal{O}(R_a^{-1})$, and use V.45 and 46):

$$k_{minLWP} = \frac{u_{*a}}{19\nu_a}$$

This can be rewritten in terms of the Reynolds number :

$$R_{amax} = 19 \qquad \text{VI.7}$$

It can be seen however that this maximum value for R_a is very sensitive to the choice of the height where the Airy-function is taken to vanish and to the choice of the height where the LWP and the exact profile are said to deviate substantially. Therefore I shall formulate VI.7 less strict and state, somewhat arbitrarily, that the condition for the LWP-model to be valid is, depending on the accuracy demanded,

$$R_a \leqslant 15 - 25 \qquad \text{VI.8}$$

There is however another condition for V.62 to make sense.
First the LWP-model must be valid, but second $|\varepsilon|$ must be small.
This latter condition results from the fact that V.62 is an expansion in $|\varepsilon| = R_a^{-2/3}$. This leads to the condition :

$$R_a \geqslant 20 \qquad \text{VI.9}$$

Thus the conditons for the validity of V.62 are given by VI.8 and 9 (and VI.4 actually, but I suppose VI.4 to be automatically fulfilled when VI.8 and 9 are fulfilled). It is seen that a range of validity of V.62 exists only if the accuracy demanded is not too large. Then this range is given by :

$$20 \leqslant R_a \leqslant 25 \qquad \text{VI.10}$$

In table 4 this range is shown for several wind speeds. By looking at this table and fig.5 it is seen that in the estimated range of validity of V.62 the analytical results coincide well with the numerical results obtained by Kawai. It may be noted that in the estimation of this range of validity $y_1 = 5\frac{\nu_a}{u_{*a}}$ is used, in accordance with the choice made by Kawai. If the thickness of the viscous sublayer is taken to be more the range of validity of V.62 increases.

u_{*a} in m/s	k_{min} in m^{-1}	k_{max} in m^{-1}
0.136	360	450
0.170	450	570
0.214	570	700
0.248	660	830

table 4. The range of validity of V.62 as determined by VI.10.

§VI.3 Two viscid models.

Miles in his viscid theory dealt with the same equations as handled in this paper though he dealt with a different situation. His interest lay with gravity waves; in this paper gravity-capillary waves are considered. Miles also found a first order correction in δ on the phase speed, though it is defined in a different formalism from the one used in the present study.
Translated Miles' result reads (Miles, '59; I have made minor approximations in the translation) :

$$c^{(1)} = \tfrac{1}{2}\,(\varphi_1' w_o + U_a') - \tfrac{1}{2} w_o\,(1 - \varphi_1' - \frac{U_a'}{w_o})^2\, e^{\frac{1}{4}i\pi} R_a^{-\frac{1}{2}} + \mathcal{O}(\frac{u_{*a}^2}{w_o^2},\ R_a^{-\frac{1}{2}} R_w^{-\frac{1}{2}}) \qquad \text{VI.11}$$

φ_1 is the inviscid solution of §V.5.
The expression derived in §V.4 for $c^{(1)}$ is :

$$c^{(1)} = \frac{1}{(2 + \frac{U_o k_o}{w_o k})}\left[\varphi_a' w_o + U_a' - \frac{g}{k w_o}\right] +$$

$$+ \frac{w_o}{(2 + \frac{U_o k_o}{w_o k})}\ \frac{i}{R_a}\left[3\varphi_a' + \varphi_a'' - \varphi_a'''\right] + \mathcal{O}(R_w^{-1},\ \varphi_a'' R_w^{-\frac{1}{2}}) \qquad \text{VI.12}$$

These two expressions are distinctly different. The viscous solution in air does not enter Miles' expression, while in the LWP-model this was the term dominating the growth.

Miles restricts himself to cases where (roughly)

$$S_a \leqslant 10$$
$$S_a = \frac{u_{*a}^2}{k \nu_a c} \qquad \text{VI.13}$$

This cannot be the cause of the differences however as some of the cases described by VI.12 lie close to this region : for u_{*a} = 0.136 m/s, k = 360 m^{-1} and c = 0.28 m/s $S_a \approx 12$.

Miles supposed though that when VI.13 was fulfilled automatically

$$S_a << R_a^{\frac{1}{2}} \qquad \text{VI.14}$$

would hold (Miles, '59). This implies that the inner and outer viscous sublayer are well separated (cf. §IV.2 and Miles, '59).
VI.14 does hold as a consequence of VI.13 in the situations Miles considered. Fot the gravity-capillary waves described in the present study VI.14 does not hold. For instance, in the situation described above $R_a^{\frac{1}{2}} \approx 7$. Indeed, the inner and outer viscous sublayers are clearly not separated as I use as approximation the critical height lying on the surface; $\eta_0 = o$. The fact that for waves considered by Miles VI.14 does hold while for the waves in this study it does not hold might be the cause of the difference in the two expressions governing the wave growth.

§VI.4 The inviscid case as a limiting situation.

When the limits

$$R_w \to \infty \qquad T \to 0$$

$$R_a \to \infty \qquad U_0 \to 0$$

are taken the results from the viscid theory ought to be in agreement with the inviscid theory. I shall go into two different aspects of this limit. One is the shift of the phase velocity of free waves due to the density of the air. The other aspect is the growth of gravity waves due to the wind.

When neither in water nor in air a basic flow is present the streamfunctions are in the above-mentioned limit given by :

$$\varphi_a = e^{-\xi} \qquad \varphi_w = e^{\xi} \qquad \text{VI.15}$$

These streamfunctions can be obtained by directly solving the Rayleigh equations (see §IV.1) for $U_a(y) = U_w(y) = 0$. The zeroth order phase velocity, describing now free gravity waves, becomes (the limit of V.37):

$$c^{(0)} = \left(\frac{g}{k}\right)^{\frac{1}{2}} \qquad \text{VI.16}$$

This is in agreement with free wave theory. The shift $c^{(1)}$ is now due solely to the density of the air as the flows have vanished.
It becomes (the limit of V.40; VI.16 has been used) :

$$c^{(1)} = \tfrac{1}{2}c^{(0)}\left[\varphi_a' - 1\right]$$

Using VI.15 this becomes

$$c^{(1)} = -c^{(0)}$$

This is in perfect agreement with Whitham, '74, p 445. The negative sign of the shift can be understood as the density of the air having the effect of a renormalization of the gravity force. Apparently the gravity becomes less when the air is present. This implies, with the use of VI.16, that the phase velocity becomes less.

When an airflow is present φ_a cannot be solved, but I suppose the derivatives of φ_a remain finite in the limit, in agreement with what is usually seen. Then Miles' result for the shift of the phase velocity, given by IV.12, is obtained exactly by taking the limit of the viscid expression V.40. The imaginary part of this shift determines the growth.

In the LWP-model the limit can also be taken. Some trouble arises as the shear in the airprofile is proportional to the Reynolds number. This implies that the shear at the interface becomes infinite and this renders the growth infinite (cf. V.62). If, to make comparison possible, the shear is taken independent of R_a the growth becomes zero in the inviscid limit (φ_a' and φ_a'' as well as φ_a''' are proportional to V). This is in agreement with the Miles' theory, which says there can be no growth when the curvature of the windprofile at the critical height is zero.

This makes it clear that the simple relation between curvature at the critical height and growth is valid only for certain waves, that is, valid only when viscosity and the viscous sublayer can be neglected. This is why Kawai's measurements are not in disagreement with the Miles' theory, though at first sight they were (see the discussion at the beginning of chapter V).

This leads to an attempt to make the boundary of the region of validity of the inviscid theory from Miles more distinct. I propose as necessary conditions for the inviscid theory :

1) the critical height must lie above the viscous sublayer.
2) the Reynolds number in air $\frac{c}{\nu_a k}$ must be 1000 or more.

To determine a maximum wavenumber from 1) I suppose that I am dealing with gravity waves and that the flow in the water can be neglected. Then the phase velocity is (VI.16) $\sqrt{\frac{g}{k}}$. The air speed at the top of the viscous sublayer is (cf. chapter III; for α_ν I have chosen 5 in agreement with Kawai, '79) :

$$U\,(y^1) = 5\,u_{*a}$$

This equals the phase velocity when

$$k = \frac{g}{25\ u_{*a}^2} \qquad \text{VI.17}$$

The maximum wavenumber for the validity of the inviscid theory follows from condition 2) and VI.17 :

$$k_{max} = \min\left(\frac{g}{25\ u_{*a}^2}\ ,\ \frac{5\ u_{*a}}{1000\ \nu_a} \right) \qquad \text{VI.18}$$

As an example, for $u_{*a} = 0.136$ m/s $k_{max} = 20\ m^{-1}$

The waves corresponding to this wavenumber are indeed gravity waves.

§VI.5 Growth due to the flow in the water.

In §V.3 the phase velocity is approximated by the form it would have when the density of the air is taken zero. This expression (V.37) can be further approximated if

$$\nu_w U_o k_o \,(2 - \frac{k_o}{k}) << \frac{g}{k} + \frac{Tk}{w} + (\frac{U_o k_o}{2k})^2$$

This will usually be the case. Then the expression reads

$$c^{(o)} = U_o - \frac{U_o k_o}{2k} + (\frac{g}{k} + \frac{Tk}{\rho_w} + (\frac{U_o k_o}{2k})^2)^{\frac{1}{2}} +$$

$$+ \; i_w \left[\frac{U_o k_o (2 - \frac{k_o}{k})}{(\frac{g}{k} + \frac{Tk}{\rho_w} + (\frac{U_o k_o}{2k})^2)^{\frac{1}{2}}} - 2k \right] \qquad \text{VI.18A}$$

Though this is a zeroth order expansion in δ the flow in the air has influence on it. This can be seen by using V.4 :

$$U_o k_o = \frac{\delta\, u_{*a}^{\;2}}{\nu_w}$$

The reason for this correlation is that the basic flows in air and water depend on each other through the continuity equations at the interface II.8, even if the air has zero density.
The imaginary part of VI.18A can be seen to be positive for

$$k > k_o \quad \text{and } U_o > \frac{k}{k_o} \left[\frac{\frac{g}{k} + \frac{Tk}{\rho_w}}{\frac{3}{4} - \frac{k_o}{k} + (\frac{k_o}{2k})^2} \right]^{\frac{1}{2}} \qquad \text{VI.19}$$

This would demand rather large values for the flow at the surface; for k_o= 250 m^{-1} and k= 500 m^{-1} the minimum flow would be U_{oc} = 0.85 m/s. However, when V.19 is fulfilled it is doubtful whether V.37, on which VI.19 is based, is still valid (see §VI.1).

§VI.6 Energy-transfer from wind to waves.

In this section an expression is derived governing the change of the total energy of air or water. This change is expressed in quantities that can be measured at the interface of the two fluids, like the normal pressure. The change in the energy may be manifested as an increase of the energy of the waves, of the internal energy or of the energy of the basic flow. It will be shown that for gravity-capillary waves the increase of the energy of the waves is due to the shearing stress as well as to the normal pressure. The normal pressure dominates the transfer of energy.

To derive the change of the total energy I use the equation for conservation of energy for an incompressible fluid with uniform density. It can be written as (cf. Batchelor, '81 p 157) :

$$\frac{D}{Dt}\left[\frac{1}{2}u^2 + E_{int} + \Psi\right] = \frac{\partial}{\partial x_j}\left[u_i(-p\delta_{ij} + 2\mu e_{ij})\right] + \\ + \frac{\partial}{\partial x_i}\left(k_H \frac{\partial T}{\partial x_i}\right) \qquad \text{VI.20}$$

e_{ij} is defined by II.1, E_{int} is the internal energy and Ψ is the potential for the body force; in the present case $\Psi = gy$.
The last term due to the temperature gradient is taken zero. The motion is supposed to be uniform in the z-direction and periodic in x and t. The above equation concerns the energy density $\mathcal{E}$. This density is integrated over the depth and the mean over one wavelength is taken.
By defining the energy per unit area E to be

$$E = \frac{1}{\lambda}\int_0^\lambda \int_{-d}^{\eta} \mathcal{E}\, dz\, dx$$

the following equation can be derived :

$$\frac{\partial E}{\partial t} = -\langle p\dot{\eta}\rangle - \langle (u\eta_x + v)2\mu\frac{\partial u}{\partial x}\rangle + \\ + \langle (u - v\eta_x)\mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\rangle \qquad \text{VI.21}$$

$\langle\ \rangle$ means averaging over one wavelength; all quantities at the r.h.s. are to be taken at the interface. The quantities are all continuous over the interface so they can be calculated either in air or in water. It follows from VI.21 that both the normal pressure (the first term at the r.h.s.) and the stress (the second term and third term) at the interface contribute to the change of the energy of the fluids. The derivation of VI.21 and its interpretations are due to Komen, unpublished notes.

In the present case all quantities of VI.21 can be expressed in terms of the basic flow U and the streamfunction $\psi = \varphi(y)\, e^{ik(x-ct)}$.
This implies an expansion in the wave steepness ε (cf. chapter II). The various expressions are here put together; they follow from II.9, 10, 18 and 22 :

$$\eta = 0 + \varepsilon \frac{\varphi_0}{c-U_0} e^{ik(x-ct)}$$

$$v = 0 - \varepsilon i k \varphi_0 e^{ik(x-ct)}$$

$$u = U_0 + \varepsilon k \varphi' e^{ik(x-ct)}$$

$$p = 0 + \varepsilon \rho k\left[-i\nu k(\varphi''' - \varphi') + w_0\varphi' + U'\varphi_0\right] e^{ik(x-ct)}$$

This leads to the following identities :

$$\langle p\dot{\eta}\rangle = -\varepsilon^2 \tfrac{1}{2} c\rho\varphi_0 k^2\left[\operatorname{Im}\varphi' - \tfrac{1}{R}\operatorname{Re}(\varphi''' - \varphi')\right] + O(\varepsilon^3) \qquad \text{VI.22}$$

$$\langle (u\eta_x + v)2\mu\frac{\partial u}{\partial x}\rangle = \varepsilon^2\varphi_0 k^3\mu \frac{w_0 - U_0}{w_0}\operatorname{Re}(\varphi') + O(\varepsilon^3)$$

$$\langle (u - v\eta_x)\mu(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x})\rangle = \mu k U_0 U' - \mu\varepsilon^2 \tfrac{1}{2} k^3\left[U'\frac{\varphi_0^2}{w_0} + \operatorname{Im}\varphi' \operatorname{Im}\varphi'' - \operatorname{Re}\varphi'(\operatorname{Re}\varphi'' - 1)\right] \qquad \text{VI.23}$$

The energy of the wave motion is ($E_{wave} = \langle \rho g \eta^2 + T\eta_x^2 \rangle$).

$$E_{wave} = \tfrac{1}{2}\rho_w \varepsilon^2 \frac{\varphi_0^2}{w_0^2}\left(g + \frac{Tk^2}{\rho_w}\right) + O(\varepsilon^3)$$

For small values of $U_0 k_0$ this becomes (V.37 is used) :

$$E_{wave} = \tfrac{1}{2}\rho_w \varepsilon^2 k \frac{\varphi_0^2}{w_0^2}\left(w_0 + \frac{U_0 k_0}{2k}\right)^2 + O(\varepsilon^3) \qquad \text{VI.24}$$

Substituting VI.22 - 24 in VI.21 gives the rate of change of the energy of either air or water :

$$\frac{\partial E_n}{\partial t} = \mu_n k U_0 U_n' + \mu_n \tfrac{1}{2}\varepsilon^2 k^3 \left[Re\,\varphi_n' (Re\,\varphi_n'' - 1) - Im\,\varphi_n'\, Im\,\varphi_n'' \right] + $$
$$+ E_{wave} \frac{\rho_n c k w_0^2}{\rho_w (w_0 + \frac{U_0 k_0}{2k})^2} \left\{ \frac{Im\,\varphi_n'}{\varphi_0} + \frac{1}{R_n} \left[\frac{Re\,\varphi_n'''}{\varphi_0} + \right.\right.$$
$$\left.\left. - \frac{(3c - 4U_0)}{c} \frac{Re\,\varphi_n'}{\varphi_0} - \frac{U_n'}{c} \right] \right\} + O(\varepsilon^3) \qquad n = a \text{ or } w \qquad \text{VI.25}$$

From VI.25 it can be deduced whether the normal pressure or the shearing stress or both contribute to the growth of the waves.
First of all it must be clear which part of VI.25 adds to the energy of the waves and which part to other forms of energy. The equality

$$2k\, Im\, c = \frac{1}{E_{wave}} \frac{\partial E_{wave}}{\partial t}$$

and V.41, the expression for $\beta = k\, Im c$, answer for this. It is seen that both the normal pressure and the stress take part in the energy transfer to the waves. This is in contrast to the inviscid case (see §VI.1) where the normal pressure alone causes the growth (naturally, as there is no shearing stress when there is no viscosity).

With the use of the LWP-model it was shown in §V.5 that the term proportional to $\frac{1}{R_{a0}} \varphi_a'''$ dominates the growth of gravity-capillary waves. Looking at VI.22,23 and 25 this can now be seen to imply that the normal pressure dominates the energy transfer.

The term $-\langle p\dot{\eta}\rangle$ is indeed the one usually measured to obtain growth-rates due to the wind (Plant, '82). From necessity this is not done at the interface but a little above it. In §VI.2 it was discussed that the viscid streamfunction, which dominates the term $-\langle p\dot{\eta}\rangle$, falls rapidly to zero above the surface. This means that the measurements will have to be done very close to the surface to be able to see anyting of the dominating term of $-\langle p\dot{\eta}\rangle$. The height would have to be less than, say,

$$Y_{max} = \frac{1}{k} R_a^{2/3}$$

y_{max} is chosen such that the Airy-function, which characterizes the viscid streamfunction, is at this height about one third of what it is at the surface. For a typical case of gravity-capillary waves y_{max} = 2 cm.

VII Conclusions.

- An analytical description of the initial stage of wave growth due to the wind based on instability of shear flow is possible.

- Viscosity is essential for the instability of wind over waves in the gravity-capillary region ($\lambda \sim 1$ cm). This explains why these waves grow while the critical height is within the viscous sublayer.

- A linear wind profile can be used to describe the growth of gravity-capillary waves by wind. Approximating this growth with a relative accuracy of $R_a^{-2/3}$ leads to growth-rates that agree with numerical values in the estimated range of validity of the approximation. This range is $20 \leq \frac{u_{*a}}{\nu_a k} \leq 25$.

- Both the normal pressure and the shearing stress at the surface contribute to the energy transfer from wind to gravity-capillary waves. The term proportional to the normal pressure - $\langle p\dot{\eta}\rangle$ -dominates this transfer.

- The inviscid theory of Miles concerning the instability of shear flow is valid only when the wind velocity at the top of the viscous sublayer is less than the phase velocity. This argument leads to a maximum wavenumber for the Miles' theory to be valid of 5 - 20 m^{-1}, depending on the airflow. For comparison, the validity of the expression for the growth in the viscid theory as derived in the present study is $k > 120$ - 500 m^{-1}.

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